Statistics 523, Problem Set 8

Wellner; 5/22/2013

Reading: Durrett, *Stochastic Calculus*; Chapter 2, pages 52-93.

Due: Friday, June 7, 2013.

1. Durrett, Lemma (c), pages 46-47: write out the details of this proof to a sufficient degree of detail to understand how the notation works. The expanded proof should probably be about twice as long as the one in the book.

2. Durrett, Exercise 3.6, page 52: If X_t is a bounded martingale, then $X_t^2 - \langle X \rangle_t$ is a uniformly integrable martingale.

3. Durrett, Exercises 3.8 & 3.9, page 52: If $S \le T$ are stopping times and $\langle X \rangle_S = \langle X \rangle_T$, then X is constant on $[S, T]$.
   Conversely, if $S \le T$ are stopping times and X is constant on $[S, T]$, then $\langle X \rangle_S = \langle X \rangle_T$.

4. Durrett, proof of 4.2.c, page 55: Durrett writes "In view of the results in the last paragraph, we can now prove the result by establishing it in the case $H = 1_{(a,b]}C$ and $K = 1_{(c,d]}$, *and* we can furthermore assume that (i) $b \le c$ or (ii) $a = c$, $b = d$." Justify these two claims.

5. Durrett, Exercise 4.2, page 56: $\|H\|_X$ is a norm.

6. Durrett, Exercise 4.3, page 56: $X \in \mathcal{M}^2$ if and only if $E(X_0^2) < \infty$ and $E\langle X \rangle_\infty < \infty$.

7. Durrett, Exercise 4.5, page 59: If X is a bounded martingale and $H \in \Pi_2(X)$, then $\|H \cdot X\|_2 = \|H\|_X$.

8. Durrett, Theorem 6.5, page 65: If X, Y are continuous local martingales, $H \in \Pi_3(X)$ and $K \in \Pi_3(Y)$, then

$$\langle H \cdot X, K \cdot Y \rangle_t = \int_0^t H_s K_s d\langle X, Y \rangle_s.$$

   Prove this theorem.

9. Durrett, Exercise 6.2, page 66: If X is a continuous local martingale and $H \in \Pi_2(X)$, then $H \cdot X \in \mathcal{M}^2$ and $\|H \cdot X\|_2 = \|H\|_X$. (Here I am not sure I believe the claim: we start with a local martingale X and end up with a martingale $H \cdot X$, at least according to Durrett's claim. Thus you may need to rephrase the claim slightly. Alternatively, give a counter-example.)

10. Durrett, Exercise 6.3, page 66: Let X be a continuous local martingale. Let $S \leq T < \infty$ be stopping times, let $C(\omega)$ be bounded with $C(\omega) \in \mathcal{F}_S$, and define $H_s = C1_{(S,T]}(s)$. Then $H \in \Pi_3(X)$ and

$$\int H_s dX_s = C(X_T - X_S).$$

11. Durrett, Exercise 6.4, page 67: If X is a continuous local martingale, then

$$\int_0^t 2X_s dX_s = X_t^2 - X_0^2 - \langle X \rangle_t.$$

12. Durrett, Exercise 6.5, page 67: Show that if X is a continuous local martingale and we evaluate at the right end point then

$$\sum_i 2X_{t_{i+1}^n}\{X(t_{i+1}^n) - X(t_i^n)\} \to_p \int_0^t 2X_s dX_s + 2\langle X \rangle_t = X_t^2 - X_0^2 + \langle X \rangle_t.$$

13. Let $f \in L_2[(0,\infty),\lambda]$ and consider the process $Z(t) \equiv \exp(\int_0^t f(s)dB(s) - \frac{1}{2}\int_0^t f^2(s)ds)$ where B is standard Brownian motion. (a) Compute $E\{Z(t)|\mathcal{F}_s\}$ where $\mathcal{F}_s$ is the $\sigma-$field generated by $\{B_r : r \leq s\}$. Is $\{Z_t\}$ a (local-) martingale?
(b) When $f(t) = \mu \neq 0$, the process $Z(t) = \exp(\mu B(t) - (1/2)\mu^2 t)$ yields the Radon-Nikodym derivative $dP_{\mu,t}/dP_{0,t}$ of $P_\mu|_{\mathcal{C}_t}$ with respect to $P_0|_{\mathcal{C}_t}$ where P_μ is the law of Brownian motion with drift, $B_\mu(t) = B(t) + \mu t$, on $(C[0,\infty), \mathcal{C}_{[0,\infty)})$; recall problem 4 of Problem set 6. Does Z in part (a) have a similar interpretation for a general L_2 function f? [See Durrett, Theorem 3.7 and Section 5.5.]

14. (Sources for harmonic functions) A fact from complex variables is that the real and imaginary parts of an analytic function satisfy the Cauchy-Riemann equations; that is, if $f(z)$ is a differentiable function of $z =$

$x + iy$ and if we write $f(x + iy) = u(x, y) + iv(x, y)$, then

$$\partial u/\partial x = \partial v/\partial y \quad \text{and} \quad \partial u/\partial y = -\partial v/\partial x.$$

(a) Use the equations in the last display to show that $u(x, y)$ and $v(x, y)$ are harmonic. What harmonic functions do you obtain from the real and imaginary parts of the analytic functions e^z and ze^z?
(b) Consider the harmonic functions f you found in (a). What is the result of applying Ito's formula to the processes of the form $X_t = f(B_t)$ where $\underline{B}_t$ is 2-dimensional Brownian motion?
(c) Consider the family of hyperbolas given by $H_\alpha = \{(x, y) : x^2 - y^2 = \alpha\}$. What is the probability that the standard two-dimensional Brownian motion in $\mathbb{R}^2$ starting at $(2, 0)$ will hit $H(1)$ before hitting $H(5)$? Hint: consider the harmonic functions obtained from the complex function $f(z) = z^2$.