Statistics 523, Problem Set 7

Wellner; 5/15/2013

Reading: Wellner Notes; Chapter 11, pages 19-33.
Durrett, *Stochastic Calculus*, Chapter 2.

Due: Wednesday, May 22, 2013.

1. Let $X(t)$ be a continuous local martingale and let φ be a convex function. Show that $\varphi(X_t)$ is a local submartingale.

2. Let X_t be a continuous local martingale and let $R < \infty$ be a stopping time. Show that $Y_s \equiv X_{R+s}$ is a local martingale with respect to $\mathcal{G}_s \equiv \mathcal{F}_{R+s}$.

3. Show that if X is a continuous local martingale with $X_t \geq 0$ and $EX_0 < \infty$, then X_t is a supermartingale.

4. Definition: If X and Y are two continuous local martingales, define
$$\langle X, Y \rangle_t \equiv \frac{1}{4}(\langle X+Y \rangle_t - \langle X-Y \rangle_t)$$
where $\langle X \rangle_t$ is the unique continuous increasing process that makes $X_t^2 - \langle X, Y \rangle_t$ a local martingale. Show that:
(i) $\langle X+Y, Z \rangle_t = \langle X, Z \rangle_t + \langle Y, Z \rangle_t$;
(ii) $\langle X - X_0, Z \rangle_t = \langle X, Z \rangle_t$;
(iii) If $a, b \in \mathbb{R}$, then $\langle aX, bY \rangle_t = ab\langle X, Y \rangle_t$;
(iv) If $a \in \mathbb{R}$, then $\langle aX \rangle_t = a^2 \langle X, \rangle_t$.

5. **Optional bonus problem**. Suppose that Z_n is a discrete time, inhomogeneous (in space) random walk defined as follows:
$$Z_{n+1} = \begin{cases} Z_n + X_n, & \text{if } Z_n > 0, \\ Z_n + Y_n, & \text{if } Z_n \leq 0, \end{cases}$$
where the X_n's are i.i.d. as X where $P(X = 1) = \alpha$, $P(X = -\alpha/(1-\alpha)) = 1-\alpha$ and $\alpha \in (0, 1/2) \cup (1/2, 1)$; and the Y_n's are i.i.d Rademacher random variables with $P(Y = \pm 1) = 1/2$. Consider embedding the process $\mathbb{Z}_n(t) \equiv n^{-1/2} Z_{[nt]}$ in Brownian motion via the methods we developed for Skorokhod embedding. Does an embedded version of $\mathbb{Z}_n$ satisfy $\|\mathbb{Z}_n - \mathbb{Z}\| \to_p 0$ for some process $\mathbb{Z}$ related to Brownian motion?