Statistics 523, Problem Set 6

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Reading: Shorack, PfS; Chapter 13, sections 5 and 6.

Due: Wednesday, May 15, 2013.

1. Let $\mathbb{U}_n = \sqrt{n}(\mathbb{G}_n - I)$ be the uniform empirical process based on $\xi_1, \ldots, \xi_n$ i.i.d. Uniform$(0,1)$ random variables. Consider the functional $g(x) = \int_0^1 x(t)dt$.
   (a) What is the limiting distribution of $g(\mathbb{U}_n)$?
   (b) Compute $g(\mathbb{U}_n)$ explicitly in terms of the ξ_i's.
   (c) Use a standard result to find the limiting distribution of $g(\mathbb{U}_n)$ as computed in (b) in a different way. Does this result agree with what you found in (a)?

2. Consider the uniform empirical process $\mathbb{U}_n$ as in problem 1 above and let
$$g(x) = \int_0^1 \frac{x^2(t)}{t(1-t)} dt.$$
Show that $g(\mathbb{U}_n) \rightarrow_d g(\mathbb{U})$. [Note that g is not continuous with respect to $\| \cdot \|_\infty \equiv \| \cdot \|$, the uniform metric on D. Hint: for $0 < \delta < 1/2$, consider the intervals $(0, \delta]$ and $[1-\delta, 1)$ separately.]

3. Use a reflection principle to show that for $0 \le y \le x$
$$P(\sup_{0 \le s \le t} \mathbb{S}(s) \ge x,\ \mathbb{S}(t) \le y) = P(\mathbb{S}(t) \ge 2x - y)\,,$$
and use this to show that the joint density of $M^+ \equiv \sup_{0 \le s \le t} \mathbb{S}(s), \mathbb{S}(t)$ is given by
$$f(x,y) = \sqrt{\frac{2}{\pi t^3}}(2x - y) \exp\left(-\frac{(2x-y)^2}{2t}\right) \qquad \text{for } \ 0 \le y \le x\,.$$

4. Suppose that $\mathbb{S}$ is standard Brownian motion on $(C[0,\infty), \mathcal{C}_{[0,\infty)})$, and let its distribution be denoted by $P = P_0$. Let $\mathbb{S}_\mu(t) \equiv \mathbb{S}(t) + \mu t$ be Brownian motion with drift μ, and let P_μ denote the distribution of $\mathbb{S}_\mu$

on $(C[0,\infty), \mathcal{C}_{[0,\infty)})$. Set $Y(t) \equiv \exp(\mu \mathbb{S}(t) - \mu^2 t/2)$. For $t > 0$ let $P_{0,t}$ and $P_{\mu,t}$ denote the distributions P_0 and P_μ restricted to $\mathcal{A}_t \equiv \{\mathbb{S}(s) : s \le t\}$. Show that the Radon - Nikodym derivative $dP_{\mu,t}/dP_{0,t} = Y(t)$.

5. Suppose that $\mathbb{S}_\mu$ is Brownian motion with drift $\mu > 0$ as in problem 2, and let $\tau \equiv \inf\{t > 0 : \mathbb{S}_\mu(t) = a\}$, $a > 0$. Use the result of problem 2 together with results from class concerning the distribution of τ when $\mu = 0$ to find the distribution of τ when $\mu > 0$. You should find that

$$\begin{aligned} P_\mu(\tau > t) &= P_\mu(\mathbb{S}_\mu(s) < a,\ 0 \le s \le t) \\ &= \Phi\left(\frac{a - \mu t}{\sqrt{t}}\right) - e^{2\mu}\Phi\left(\frac{-a - \mu t}{\sqrt{t}}\right) \end{aligned}$$

and

$$f_\tau(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{(a-\mu t)^2}{2t}\right) \qquad \text{for } t \ge 0\,.$$

This is the *inverse Gaussian* density. [Note that this reduces to the density of τ_a from class when $\mu = 0$.]