Statistics 523, Problem Set 5

Wellner; May 1, 2013

Reading: Shorack, PfS Course Notes, Chapter 12, pages 326-348.
Shorack, PfS Course Notes, Chapter 8, pages 175-178.

Due: Wednesday, May 8, 2013.

Reminder: Midterm exam; Friday 10 May

1. Suppose that $\{X(t) : t \geq 0\}$ is a process with stationary and independent increments with $X(0) = 0$ and characteristic function of $X(t)$given by

$$Ee^{iuX(t)} = \exp(-tc|u|^\alpha\{1 - i\text{sign}(u)C_\alpha\})$$

where $\alpha \in (0, 1)$, $c \geq 0$ and $C_\alpha = \tan(\pi\alpha/2)$. Thus the (marginal, or one-dimensional) distributions of $X(t)$ are completely asymmetric stable laws with exponent $\alpha \in (0, 1)$.
(a) Show that $X(t) \stackrel{d}{=} t^{1/\alpha}X(1)$ for all $t > 0$.
(b) Let $0 < r < \alpha$. Use (a) to compute $E|X(t)|^r$ in terms of $E|X(1)|^r$ where the latter is finite by Problem xx of problem set 3.
(c) Apply Theorem 12.2.2 to show that $X : (\Omega, \mathcal{A}, P) \to (R_{[0,1]}, \mathcal{B}_{[0,1]}, P_X)$ has an equivalent version $Z : (\Omega, \mathcal{A}, P) \to (R_{[0,1]}, \mathcal{B}_{[0,1]}, P_Z)$ which satisfies $Z : (\Omega, \mathcal{A}, P) \to (D, \mathcal{D}, P_Z)$ with $Z(t) = X(t)$ a.s. for each $t \in [0, 1]$.
(d) Is there an analogous theorem when X is viewed as a process with values in $(R_{[0,\infty)}, \mathcal{B}_{[0,\infty)}, P_X)$ and with versions in $(D_{[0,\infty)}, \mathcal{D}_{[0,\infty)})$?

2. Now let $\mathbb{S}$ be a standard Brownian motion on $[0, \infty)$, let $X(t)$ be a completely asymmetric stable process (sometimes called a *stable subordinator*) of index $\alpha \in (0, 1)$ as in problem 1 above which is independent of $\mathbb{S}$. Consider the new process $Y(t) \equiv \mathbb{S}(X(t))$ for $t \geq 0$.
(a) Use a calculation similar to that of problem 4, Problem set 3, to show that Y is a symmetric stable process of index 2α.
(b) Does Y have stationary independent increments?

3. Let $\tau = \tau_{ab}$ of Theorem 12.6.1; i.e. $\tau_{a,b} = \inf\{t : \mathbb{S}(t) \in (-a, b)^c\}$ for $a, b > 0$. Show that $E(\tau^2) \leq 4ab(a + b)^2$. (This is slightly different from the statement of (4) in Theorem 12.6.1 on page 319, but

seems to be consistent with (5) on the same page with $r = 2$. My current computation yields, in fact, $E(\tau^2) \le 2ab(a+b)^2$. *Hint:* use the martingale $\mathbb{S}^4(t) - 6t\mathbb{S}^2(t) + 3t^2$ (which follows from considering the 4th derivative with respect to θ of the exponential martingale $V_\theta(t) = \exp(\theta\mathbb{S}(t) - \theta^2 t^2/2)$ at $\theta = 0$; see Exercise 12.7.3, page 325.

4. PfS Course Notes, Exercise 11.1.1, page 274: 12.10.1 page 338: Suppose that $Y_1, Y_2, \ldots$ are i.i.d. Exponential(1) random variables. Set $\eta_{n,j} \equiv \sum_{i=1}^{j} Y_i / \sum_{i=1}^{n+1} Y_i$ and let $\eta_n \equiv (\eta_{n,1}, \ldots, \eta_{n,1})$. Show that $\eta_n \stackrel{d}{=} (\xi_{n:1}, \ldots, \xi_{n:n})$ where $0 \le \xi_{n:1} \le \cdots \le \xi_{n:n} \le 1$ are the order statistics of $\xi_1, \ldots, \xi_n$ i.i.d. Uniform$(0,1)$ random variables.

5. **Optional bonus problem:** Exercise 12.7.1, PfS Course notes, page 323. That is, prove that

$$P(\|\mathbb{S}\|_0^1 > a) \;=\; 1 - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2\pi^2}{8a^2}\right).$$

See Chung (1974), page 223. This yields the “small ball probability”

$$\begin{aligned} P(\|\mathbb{S}\|_0^1 \le a) \;&=\; \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2\pi^2}{8a^2}\right) \\ &\le\; \frac{4}{\pi}\exp(-\pi^2/(8a^2)) \end{aligned}$$

which converges to zero exponentially fast as $a \to 0$, and is a general phenomena associated with Gaussian processes: see Ledoux and Talagrand (1991), pages 60-61 and 289-290.