Statistics 523, Problem Set 4

Wellner; 4/24/2013

Reading: Shorack, PfS; Chapter 12, pages 301 - 333;

Due: Wednesday, May 1, 2013.

1. PfS Course Notes, Exercise 12.3.1, page 309:
   Let $Z \sim N(0,1)$ and the Brownian bridges $\mathbb{V}$, $\mathbb{U}^{(1)}$, and $\mathbb{U}^{(2)}$ be independent. Fix $a > 0$. Show that:
   (11) $\mathbb{S}(t) = \mathbb{V}(t) + tZ$, $0 \le t \le 1$ is a Brownian motion.
   (12) $\mathbb{S}(at)/\sqrt{a}$, $0 \le t < \infty$ is a Brownian motion.
   (13) $\mathbb{S}(t+a) - \mathbb{S}(a)$, $t \ge 0$, is a Brownian motion.
   (14) $\sqrt{1-a}\mathbb{U}^{(1)} \pm \sqrt{a}\mathbb{U}^{(2)}$ is a Brownian bridge if $0 \le a \le 1$.
   (15) $\mathbb{Z}(t) \equiv \{\mathbb{U}^{(1)}(t) + \mathbb{U}^{(2)}(1-t)\}/\sqrt{2}$, $0 \le t \le 1/2$ is a Brownian bridge.
   (16) $\mathbb{U}(t) = (1-t)\mathbb{S}(t/(1-t))$, $0 \le t \le 1$, is a Brownian bridge; use the LIL at infinity to show that this $\mathbb{U}$ converges to 0 at $t = 1$.
   (17) $t\mathbb{S}(1/t)$, $0 \le t < \infty$ is a Brownian motion; apply the LIL of (10) to verify that these sample paths converge to 0 at $t = 0$.

2. PfS Course Notes, Exercise 12.3.6, page 310:
   (a) Suppose that h and $\tilde{h}$ on $[0,1]$ are in $L_2[0,1]$. View *white noise* as an operator $d\mathbb{S}$ that takes the function h into a random variable $\int_{[0,1]} h(t)d\mathbb{S}(t)$ in the sense of $\rightarrow_{L_2}$. Define this integral first for step functions, and the use Exercise 4.4.5 to define it in general. Then show that $\int_{[0,1]} h(t)d\mathbb{S}(t)$ exists as such an L_2-limit for all $h \in L_2[0,1]$.
   (b) In case h has a bounded derivative h'on $[0,1]$, show that
   $$\int_{[0,1]} h(t)d\mathbb{S}(t) = h\mathbb{S}\Big|_{0+}^{1-} - \int_{[0,1]} \mathbb{S}(t)h'(t)dt.$$
   (c) Determine the joint distribution of $\int_{[0,1]} h(t)d\mathbb{S}(t)$ and $\int_{[0,1]} \tilde{h}(t)d\mathbb{S}(t)$.

3. PfS Course Notes, Exercise 12.9.2, page 332:
Let $V_n(r) \equiv \sum_{k=1}^{n} |\mathbb{S}(t_{nk}) - \mathbb{S}(t_{n,k-1})|^r$ and $\|\mathcal{P}_n\| \equiv \sup_{1\le k\le n} |t_{nk} - t_{n,k-1}|$.
(α) Let $Z \sim N(0,1)$. Let $r > 0$. Show that:
(a) $C_r \equiv E|Z|^r = 2^{r/2}\Gamma((r+1)/2)/\sqrt{\pi}$.
(b) $E|\mathbb{S}(t_{n,k-1}, t_{n,k}]|^r = C_r |t_{n,k} - t_{n,k-1}|^{r/2}$,
$Var(|\mathbb{S}(t_{n,k-1}, t_{n,k}]|^r) = (C_{2r} - C_r^2)|t_{n,k} - t_{n,k-1}|^r$.
(β) Now show that $EV_n(2) = 1$ and $Var(V_n(2)) \le (C_{2r} - C_r^2)\|\mathcal{P}_n\|$ and hence

$$\sum_{n=1}^{\infty} P(|V_n(2) - 1| \ge \epsilon) \le \epsilon^{-2}(C_{2r} - C_r^2)\sum_{n=1}^{\infty} \|\mathcal{P}_n\|.$$

(γ) Finally prove that $V_n(2) \to_{a.s.} 1$ if $\sum_{n=1}^{\infty} \|\mathcal{P}_n\| < \infty$.

4. PfS Course Notes, Exercise 12.9.3, page 332: Prove theorem 9.1(b) when all $t_{nk} = k/2^n$. That is, with $V_n(r) \equiv \sum_{k=1}^{n} |\mathbb{S}(t_{nk}) - \mathbb{S}(t_{n,k-1})|^r$, show that $V_n(1) \to_{a.s.} \infty$ if $\|\mathcal{P}_n\| \equiv \sup_{1\le k\le n} |t_{nk} - t_{n,k-1}| \to 0$.

5. Let $\mathbb{S}$ denote a standard Brownian motion process on $[0,\infty)$, and define

$$\mathbb{X}(t) = e^{-t}\mathbb{S}(e^{2t}) \quad \text{for} \quad t \in (-\infty, \infty).$$

(a) Compute $E\mathbb{X}(t)\mathbb{X}(s)$ and $E(\mathbb{X}(t)\mathbb{X}(t+h))$.
(b) Is $\mathbb{X}$ a Gaussian process?
(c) Is $\mathbb{X}$ a stationary process?