Statistics 523, Problem Set 3

Wellner; 4/17/2013

Reading: Shorack, PfS Course Notes, Chapter 11, pages 281-287
Shorack, PfS Course Notes, Chapter 12, pages 301-325.

Due: Wednesday, April 24, 2013.

1. (a) Give an example of a random variable Y with distribution function F on $\mathbb{R}^+ = [0, \infty)$ for which $EY^r = \infty$ for all $r > 0$.
(b) Does your example in (a) have $Eg(Y) < \infty$ for some measurable function g with $g(y) \to \infty$ as $y \to \infty$.

2. Let $X_1, X_2, \ldots$ be i.i.d. with a density function f that is symmetric about 0 and continuous and positive at 0. Show that

$$\frac{1}{n}\left(\frac{1}{X_1} + \cdots + \frac{1}{X_n}\right) \to_d Y$$

where Y has a Cauchy distribution.

3. (a) Let Y be a stable random variable with $\theta = 1$ and $0 < \alpha < 1$. Show that $P(Y \ge 0) = 1$. (b) Let Y be as in (a). By the conclusion of (a) the Laplace transform of Y, $\psi(\lambda) = E\exp(-\lambda Y)$ is well-defined. Show that $Y_1 + \cdots + Y_k \stackrel{d}{=} a_k Y + b_k$ holds with $b_k = 0$ (and $Y_1, \ldots, Y_k$ i.i.d. as Y).
(c) Show that $\psi(\lambda)^n = \psi(n^{1/\alpha}\lambda)$ and hence that $\psi(\lambda) = \exp(-c\lambda^\alpha)$ for some $c > 0$.

4. Show that if X is symmetric stable with index α and $Y \ge 0$ is an independent stable random variable with index $\beta < 1$, then $XY^{1/\alpha}$ is symmetric stable with index $\alpha\beta$.

5. Find a random variable Y with distribution function F having $EY^2 = \infty$ but with $F \in \mathcal{D}(Normal)$.

6. **Optional bonus problem**. What happens in problem 2 when the hypothesis of symmetry is dropped?

7. **Optional bonus problem**. In the notation of Darling (1952), suppose that $F \in \mathcal{D}(G_1)$; i.e a stable distribution with $\alpha = 1$. What is the characteristic function of the limiting distribution of $S_n/X_n^* = (\sum_{i=1}^n X_i)/(\max_{i \le n} X_i)$ where $X_1, \ldots, X_n, \ldots$ are i.i.d. as F?