Statistics 523, Problem Set 2

Wellner; 4/10/13

Reading: Shorack, PfS Course Notes, Chapter 11, pages 281-285.

Due: Wednesday, April 17, 2013.

1. PfS Course Notes, Exercise 11.1.1, page 274: (Chf expansions for the uan array $X_{n1}, \ldots, X_{nn}$) Consider a uan array of rv's X_{nk}.
(a) Let F_{nk} and ϕ_{nk} denote the df and the chf of X_{nk}. Show that
$$\max_{k \le n} |\phi_{nk}(t) - 1| \to 0 \ \text{ uniformly on every finite interval.}$$
(b) Set $\epsilon_n(t) \equiv \sum_{k=1}^n |\phi_{nk}(t) - 1|^2$. Show that if $X_{nk} \sim (0, \sigma_{nk}^2)$ and $\sigma_n^2 \equiv \sum_{k=1}^n \sigma_{nk}^2 \le M < \infty$ with $\max_{k \le n} \sigma_{nk}^2 \to 0$, then $\epsilon_n(t) \to 0$ uniformly in t on each finite interval.

2. PfS Course Notes, Exercise 11.1.2, page 274:
(a) If ϕ is id, then $\phi(t) \ne 0$ for any t.
(b) Let ϕ and ϕ_n denote the chf of Y and of the Y_{nk}'s respectively where $Y \stackrel{d}{=} Y_{n1} + \cdots + Y_{nn}$ and the Y_{nk}'s are i.i.d. for each n. Show that these Y_{nk}'s form a uan array.
(c) If $Y_m \to_d Y$ for id random variables Y_m, then Y is id.

3. PfS Course Notes, Exercise 11.2.1, page 283: Suppose that $a_n \nearrow$ with $a_1 = 1$ and suppose that $a_{mk} = a_m a_k$ for all $k, m \ge 1$. Then show that $a_n = n^{1/\alpha}$ for some $\alpha \ge 0$.

4. PfS Course Notes, Exercise 11.2.2, page 283: Suppose that $Y \sim G$ is stable with characteristic exponent α. Then $E|Y|^r < \infty$ for all $0 < r < \alpha$. (Hint: use the inequalities of section 8.3 to show that $nP(|Y| > a_n x)$ is bounded in n where $a_n \equiv n^{1/\alpha}$, and then bound the appropriate integral.

5. Let $\gamma \ge 0$, $\delta \in \mathbb{R}$, $\delta_j \in \mathbb{R}$, and suppose that $0 < \gamma + \sum_{j=1}^\infty \delta_j^2 < \infty$. Define, for $t \in \mathbb{R}$,
$$\Psi(t) = e^{-\gamma t^2 + \delta t} \prod_{j=1}^\infty (1 + \delta_j t) e^{-\delta_j t},$$
and consider $\phi(v) \equiv 1/\Psi(iv)$.
(a) Is ϕ the characteristic function of some random variable Y? If so, identify Y in terms of some simpler independent random variables.
(b) Is Y infinitely divisible?

6. **Optional bonus problem:** What is the Lévy measure in the representation of the characteristic function of Y?
(Remarks: This problem is related to Schoenberg's (1952) characterization of Polya frequency functions of order infinity, and is intimately related to issues involving log-concavity.)