**Statistics 523, Problem Set 1**

Wellner; 4/3/13

**Reading:** Shorack, PfS Course Notes, Chapter 10, pages 225-228;
Shorack, PfS Course Notes, Chapter 11, pages 273-280.

**Due:** Wednesday, April 10, 2013.

1. PfS Course Notes, Exercise 10.1.5, page 228. (Empirical process, Doob) Let $\mathbb{U}_n \equiv \sqrt{n}(\mathbb{G}_n - I)$ be the uniform empirical process; here $\mathbb{G}_n(t) \equiv n^{-1}\sum_{i=1}^n 1_{[0,t]}(\xi_i)$ where $\xi_i$ are i.i.d. Uniform$(0,1)$ random variables and $I(t) = t$ for $0 \le t \le 1$. Let $\mathbb{U}$ denote a standard Brownian bridge process on $[0,1]$. Show that $\mathbb{U}_n \to_{fd} \mathbb{U}$ as $n \to \infty$; that is, show that for any set of points $0 < t_1 < t_2 < \cdots < t_k < 1$ we have
$$(\mathbb{U}_n(t_1), \ldots, \mathbb{U}_n(t_k)) \to_d (\mathbb{U}(t_1), \ldots, \mathbb{U}(t_k)) \quad \text{as} \quad n \to \infty.$$

2. PfS Course Notes, Exercise 10.1.6, page 228. (Partial sum process of i.i.d. random variables) Let $\mathbb{S}_n$ denote the partial sum process of i.i.d. $(0,1)$ random variables $X_i$ (that is, $E(X_i) = 0$ and $Var(X_i) = 1$: thus $\mathbb{S}_n(t) \equiv n^{-1/2}\sum_{i=1}^{[nt]} X_i$ for $0 \le t \le 1$. Let $\mathbb{S}$ denote standard Brownian motion on $[0,1]$. Show that $\mathbb{S}_n \to_{f.d.} \mathbb{S}$.

3. PfS Course Notes, Exercise 10.2.1, page 236. (Characterization of "uan") Show that the following are equivalent:
(a) $|X_{n,k}|$'s are uan; that is, $\max_{1\le k\le n} P(|X_{n,k}| \ge \epsilon) \to 0$ for all $\epsilon > 0$.
(b) $\max_{1\le k\le n} |\phi_{nk}(t) - 1| \to 0$ uniformly on every finite interval of $t$'s.
(c) $\max_{1\le k\le n} E(X_{n,k}^2 \wedge 1) = \max_{1\le k\le n} \int (x^2 \wedge 1) dF_{nk}(x) \to 0$.

4. PfS Course Notes, Exercise 10.2.8, page 237.
(i) Show that Lindeberg's condition that $LF_n(\epsilon) \to 0$ for all $\epsilon > 0$ implies Feller's condition that $\max_{1\le k\le n} \sigma_{n,k}^2/\sigma_n^2 \to 0$.
(ii) Let $X_{n1}, \ldots, X_{nn}$ be row independent Poisson$(\lambda/n)$ random variables with $\lambda > 0$. Discuss which of the Lindeberg-Feller, Liapunov, and Feller conditions holds in this context. [The Liapunov $(2+\delta)$ condition is as follows: for some $0 < \delta \le 1$ we have
$$\sum_{k=1}^n E|X_{nk} - \mu_{nk}|^{2+\delta}/\sigma_n^{2+\delta} \to 0.]$$
(iii) Repeat part (ii) when $X_{n1}, \ldots, X_{nn}$ are row independent and all have the probability density $cx^{-3}(\log x)^2$ on $x \ge 3$ (for some constant $c > 0$).

(iv) Repeat part (ii) when $P(X_{nk} = a_k) = P(X_{nk} = -a_k) = 1/2$ for row-independent $X_{nk}$'s. Discuss this for general $a_k$'s and present two or three interesting examples for which the various conditions differ (i.e. hold or fail to hold).

5. PfS Course Notes, Exercise 10.1.10, page 230. (Special cases of Gnedenko's theorem) Let $X_{n:n}$ be the maximum of an i.i.d. sample $X_1, \ldots, X_n$ from $F$. Then

   **(a)** If $1 - F(x) = e^{-x}$ for $x \geq 0$, then $P(X_{n:n} - \log n \leq y) \to e^{-e^{-y}}$ for all $y \in \mathbb{R}$.

   **(b)** If $1 - F(x) = |x|^b$ for $-1 \leq x \leq 0$ with $b > 0$, then $P(n^{1/b} X_{n:n} \leq y) \to \exp(-|y|^b)$ for all $y < 0$.

   **(c)** If $1 - F(x) = 1/x^a$ for $x \geq 1$ with $a > 0$, then $P(X_{n:n}/n^{1/a} \leq y) \to \exp(-y^{-a})$ for all $y > 0$.

6. (Optional bonus problem 1). For each of the three parts in the previous problem, show that the densities (of the distributions on the left which converge in law) exist and converge pointwise (to the densities of the distributions on the right). Use this to show that the convergence actually occurs in the sense of convergence in total variation distance. Can you quantify the rate of convergence?

7. (Optional bonus problem 2). Construct a proof of (10) implies (11) in the Lindeberg-Feller CLT without using characteristic functions.