Statistics 523, Problem Set 6

Wellner; 5/7/2010

Reading: Shorack, PfS; Chapter 12, pages 247 - 283.

Due: Friday, May 14, 2010.

1. PfS, Exercise 13.7.1, page 382: Show that $EZ_\tau^{(2)} = 0$ and $EZ_\tau = 1$ where $S_n = Y_1 + \cdots + Y_n$, $Y_i = 2X_i - 1$, $X_i \sim \text{Bernoulli}(p)$ are i.i.d.,

$$\tau \equiv \inf\{n : S_n = -a \text{ or } b\}$$

with $a, b \in \mathbb{N}$, and $Z_n^{(2)} \equiv S_n - n(p-q)$, $Z_n \equiv (q/p)^{S_n}$.

Solution: Now $\tau \wedge m \nearrow \tau$ as $m \to \infty$ while S_τ takes on the values $-a$ or b. By the simple optional sampling theorem applied to the mean zero martingale $\{Z_n^{(2)}$ and the bounded stopping times $\tau \wedge m$ it follows that $EZ_{\tau\wedge m}^{(2)} = 0$ for all m. Thus, using linearity of expectation

$$\begin{aligned} 0 &= \lim_m 0 = \lim_m EZ_{\tau\wedge m}^{(2)} = \lim_m E\{S_{\tau\wedge m} - (\tau\wedge m)(p-q)\} \\ &= \lim_m ES_{\tau\wedge m} - \lim_m (\tau \wedge m)(p-q) \\ &= ES_\tau - E(\tau)(p-q) = E\{S_\tau - \tau(p-q)\} = EZ_\tau^{(2)} \end{aligned}$$

where we using the dominated convergence theorem on the first term and the monotone convergence theorem on the second term.

Similarly, by the simple optional sampling theorem applied to the mean 1 martingale $\{Z_n\}$ and the bounded stopping times $\tau\wedge m$ it follows that $EZ_{\tau\wedge m} = 1$ for all m. Thus

$$\begin{aligned} 1 &= \lim_m 1 = \lim_m EZ_{\tau\wedge m} = \lim_m E(q/p)^{S_{\tau\wedge m}} \\ &= E(q/p)^{S_\tau} = EZ_\tau \end{aligned}$$

by the dominated convergence theorem since $(q/p)^{S_{\tau\wedge m}} \to_{a.s.} (q/p)^{S_\tau}$ as $m \to \infty$ and $(q/p)^{S_{\tau\wedge m}} \le (q/p)^{-a} \vee (q/p)^b$ for all m.

Another way of proving that the optional sampling theorem applies to τ in this case is to proceed via problem 3 below: note that for each $n \ge 0$

$$P(\tau \le n + a + b|\mathcal{A}_n) \ge p^{b-S_n} + q^{S_n+a} \ge (p\wedge q)^{(a+b)} \equiv \epsilon > 0$$

since $p \in (0,1)$. Thus the hypothesis of problem 3 holds with $N = a+b$ and $\epsilon \equiv (p \wedge q)^{a+b}$. It follows from problem 3 that $E\tau < \infty$. From this together with the fact that the conditional expectations of the increments of $Z_n^{(2)}$ and Z_n are a.s. bounded implies that the sufficient conditions of (11) of the corollary 13.6.1 to the optional sampling theorem are in force.

2. PfS, Exercise 13.7.2, page 382. Suppose that $\mathbb{S}_\mu$ is Brownian motion with drift: $\mathbb{S}_\mu(t) = \mathbb{S}(t) + \mu t$ for $t \geq 0$. Define the stopping time $\tau_{a,b} \equiv \tau \equiv \inf\{t \geq 0 : \ \mathbb{S}_\mu(t) = -a \text{ or } b\}$ where $-a < 0 < b$. Here

$$\mathbb{S}_0(t), \quad \mathbb{S}_0^2(t) - t, \quad \mathbb{S}_\mu(t) - \mu t \quad \text{ are mean } \ 0 \text{ martingales}, \tag{1}$$

while

$$\exp(\theta(\mathbb{S}_\mu(t) - \mu t) - \theta^2 t/2) \quad \text{is a mean } \ 1 \text{ martingale for every } \theta \in \mathbb{R}. \tag{2}$$

By the optional sampling theorem applied to the martingales in the last two displays, show that

$$\begin{array}{ll} P(\mathbb{S}(\tau) = -a) = b/(a+b), & \text{if } \mu = 0, \\ E\tau = ab, & \text{if } \mu = 0, \\ P(\mathbb{S}_\mu(\tau) = -a) = \frac{1-e^{2\mu b}}{1-e^{2\mu(a+b)}}, & \text{if } \mu \neq 0, \\ E\tau = \frac{b}{\mu} - \frac{a+b}{\mu}\frac{1-e^{2\mu b}}{1-e^{2\mu(a+b)}}, & \text{if } \mu \neq 0. \end{array}$$

Furthermore, if $\mu < 0$, then

$$P(\sup_{t \geq 0} \mathbb{S}_\mu(t) \geq b) = \exp(-2|\mu| b) \ \text{ for all } b > 0;$$

i.e. $\sup_{t \geq 0} \mathbb{S}_\mu(t) \sim \text{Exponential}(2|\mu|)$.

Solution: By first considering the bounded stopping times $\tau \wedge m$ and the martingales in (1) and (2), it follows from the simple optional sampling theorem and limiting arguments as in the discrete case that

$$\begin{array}{l} 0 = E\mathbb{S}(\tau) = -aP(\mathbb{S}(\tau) = -a) + b(1 - P(\mathbb{S}(\tau) = a)), \\ 0 = E\{\mathbb{S}^2(\tau) - \tau\}, \\ 0 = E\{\mathbb{S}_\mu(\tau) - \mu\tau\}, \\ 1 = E\exp(\theta(\mathbb{S}_\mu(\tau) - \mu\tau) - \theta^2\tau/2) \ \text{ for every } \quad \theta \in \mathbb{R} \end{array}$$

The first relation in the last display implies that $P(\mathbb{S}(\tau) = -a) = b/(a+b)$, and then the second relation yields

$$E(\tau) = E\mathbb{S}^2(\tau) = a^2 \frac{b}{b+a} + b^2 \frac{a}{b+a} = ab.$$

Choosing $\theta = -2\mu$, the last relation involving the exponential martingale gives

$$1 = E\exp(-2\mu\mathbb{S}(\tau) - (2\mu^2)\tau) = E\exp(-2\mu(\mathbb{S}(\tau) + \mu\tau)) = E\exp(-2\mu\mathbb{S}_\mu(\tau)),$$

and hence

$$1 = \exp(-2\mu(-a))P(\mathbb{S}_\mu(\tau) = -a) + \exp(-2\mu b)(1 - P(\mathbb{S}_\mu(\tau) = -a)).$$

Solving this for $P(\mathbb{S}_\mu(\tau) = -a)$ yields

$$\begin{aligned} P(\mathbb{S}_\mu(\tau) = -a) &= \frac{1 - e^{-2\mu b}}{e^{-2\mu(-a)} - e^{-2\mu b}} \\ &= \frac{1 - e^{-2\mu b}}{e^{-2\mu(-a)} - e^{-2\mu b}} \cdot \frac{e^{2\mu b}}{e^{2\mu b}} \\ &= \frac{e^{2\mu b} - 1}{e^{2\mu(a+b)} - 1} = \frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}} \qquad (3) \end{aligned}$$

as claimed. Using this together with the third relation gives

$$\begin{aligned} E\tau &= \frac{1}{\mu} E\mathbb{S}_\mu(\tau) \\ &= \frac{1}{\mu}\{b(1 - P(\mathbb{S}_\mu(\tau) = -a)) + (-a)P(\mathbb{S}_\mu(\tau) = -a)\} \\ &= \frac{b}{\mu} - \frac{b+a}{\mu} \frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}}. \end{aligned}$$

Finally, letting $a \to \infty$ in (3) yields, when $\mu < 0$,

$$P(\sup_{t \geq 0} \mathbb{S}_\mu(t) < b) = \lim_{a \to \infty} P(\mathbb{S}_\mu(\tau) = -a) = 1 - \exp(-2|\mu|b).$$

3. PwM, E10.5, page 233. Suppose that T is a stopping time such that for some $N \in \mathbb{N}$ and some $\epsilon > 0$ we have, for every n,

$$P(T \leq n + N | \mathcal{F}_n) > \epsilon \quad \text{a.s.}$$

Prove by induction using $P(T > kN) = P([T > kN] \cap [T > (k-1)N])$ that

$$P(T > kN) \leq (1-\epsilon)^k \quad \text{for} \quad k = 1, 2, \ldots,$$

and hence $E(T) < \infty$.

Solution: First, note that by applying the hypothesis with $n = 0$ we have

$$P(T > N) = EP(T > N + 0|\mathcal{F}_0) \leq E(1-\epsilon) = 1 - \epsilon,$$

so the claimed inequality holds for $k = 1$. Suppose it holds for $k-1$ for some $k \geq 2$. We want to show that it then holds for k. But since $[T > (k-1)N] = [T \leq (k-1)N]^c \in \mathcal{F}_{(k-1)N}$,

$$\begin{aligned}
P(T > kN) &= P([T > kN] \cap [T > (k-1)N]) = E\{1_{[T>kN]} 1_{[T>(k-1)N]}\} \\
&= EE\{1_{[T>kN]} 1_{[T>(k-1)N]} | \mathcal{F}_{(k-1)N}\} \\
&= E\{1_{[T>(k-1)N]} E\{1_{[T>kN]} | \mathcal{F}_{(k-1)N}\}\} \\
&= E\{1_{[T>(k-1)N]} P((T > kN | \mathcal{F}_{(k-1)N})\} \\
&= E\{1_{[T>(k-1)N]} P((T > N + (k-1)N | \mathcal{F}_{(k-1)N})\} \\
&\leq E\{1_{[T>(k-1)N]} (1-\epsilon)\} \quad \text{by the hypothesis with} \quad n = (k-1)N \\
&= (1-\epsilon) P(T > (k-1)N) \\
&\leq (1-\epsilon)(1-\epsilon)^{k-1} \quad \text{by the induction hypothesis} \\
&= (1-\epsilon)^k.
\end{aligned}$$

Thus $P(T/N > k) \leq (1-\epsilon)^k$ for $k \geq 1$, and it follows from Inequality 8.2.1 that

$$E(T/N) \leq 1 + \sum_{k=1}^{\infty} P(T/N > k) \leq 1 + \sum_{k=1}^{\infty} (1-\epsilon)^k = 1/\epsilon < \infty$$

since $\epsilon > 0$. Therefore $E(T) \leq N/\epsilon < \infty$.

4. Let $X_1, X_2, \ldots$ be i.i.d. Rademacher random variables (so that $P(X_i = \pm 1) = 1/2$), and let $S_n = \sum_{j=1}^{n}$ be the associated partial sum (or simple random walk) process. Let $\mathcal{A}_n$ be the sigma-field generated by $S_1, S_2, \ldots, S_n$. Now $\{S_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is a martingale, and $\{|S_n|, \mathcal{A}_n\}$ is a sub-martingale. Find a predictable increasing process $\{A_n\}$ so that

with $Y_n \equiv |S_n| - A_n$, $\{Y_n, \mathcal{A}_n\}$ is a martingale. You should compute A_n as explicitly as possible in terms of the X_j's.

Solution: By Doob's decomposition of a sub-martingale, we know that the a.s. unique predictable process $\{A_n\}$ that makes $|S_n| - A_n$ a martingale is given by

$$\begin{aligned} A_n &= \sum_{k=1}^{n} \{E(|S_k||\mathcal{A}_{k-1}) - |S_{k-1}|\} \\ &= \sum_{k=1}^{n} E\{|S_k| - |S_{k-1}|\big|\mathcal{A}_{k-1}\} \\ &= \sum_{k=1}^{n} E\{|S_{k-1} + X_k| - |S_{k-1}|\big|\mathcal{A}_{k-1}\}. \end{aligned}$$

Now

$$\begin{aligned} E\{|S_{k-1} + X_k| - |S_{k-1}|\big|\mathcal{A}_{k-1}\} &= \frac{1}{2}|S_{k-1}+1| + \frac{1}{2}|S_{k-1}-1| - |S_{k-1}| \\ &= 1\{S_{k-1} = 0\} \end{aligned}$$

since

$$\frac{1}{2}|y+1| + \frac{1}{2}|y-1| - |y| = \begin{cases} 0, & \text{if } y \geq 1 \\ 0, & \text{if } y \leq -1 \\ 1, & \text{if, } y = 0. \end{cases}$$

Thus it follows that

$$A_n = \sum_{k=1}^{n} 1\{S_{k-1} = 0\} = \#\{0 \leq k \leq n-1 : \ S_k = 0\}.$$

As noted in class,

$$Z_n \equiv |S_n| - A_n = \sum_{k=1}^{n} \text{sign}(S_{k-1}) X_k = \sum_{k=1}^{n} \text{sign}(S_{k-1}) \Delta S_k \tag{4}$$

gives a representation of the martingale $\{Z_n\}$ as an H-transform of the basic martingale $\{S_n\}$ with $H_k \equiv \text{sign}(S_{k-1})$. This is a discrete time or simple random walk version of Tanaka's formula for Brownian motion:

$$|B_t| - L(t, 0) = \int_0^t \text{sign}(B_s) dB_s$$

where $L(t,0)$ is the *local time* at 0 up to time t for the Brownian motion B. See e.g. Csörgő and Révész (1985).