Statistics 523, Problem Set 4 Solutions

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1. PfS, Exercise 13.3.5, page 359. Let $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ be a submg. Show that the following are equivalent:
(a) The X_n^+'s are uniformly integrable.
(b) There exists a rv Y that closes the submg.
(c) When these hold, then X_∞ closes the submg.

Solution: Suppose that (a) holds. Then for some M we have $EX_n^+ \le M < \infty$; $M = \lambda_1 + 1$ works where $\lambda_1 \equiv \inf\{\lambda > 0 : E\{X_n^+ 1_{[X_n^+ \ge \lambda]}\} \le 1\}$. It follows from the s-mg convergence theorem that $X_n \to_{a.s.} X_\infty$ where $X_\infty \in L_1$.

Now fix $a \le 0$, and consider the convex function $\varphi_a(x) = (x \vee a) + |a|$. Note that $\varphi_a(x) \ge 0$ for all x and is increasing (i.e. non-decreasing). Thus $Y_n^a \equiv \varphi_a(X_n)$ is a sub-martingale, and $Y_n^a = (Y_n^a)^+$. Moreover, uniform integrability of $\{X_n^+\}$ implies uniform integrability of $\{Y_n^a\} = \{(Y_n^a)^+\}$: note that $Y_n^a \le X_n^+ + |a|$. Thus by the s-mg convergence theorem Y_n^a is uniformly integrable, and converges a.s. and in L_1 to Y_∞^a which closes the sub-mg. But since φ_a is continuous and $X_n \to_{a.s} X_\infty$ it follows that

$$\begin{aligned} &Y_n^a = \varphi_a(X_n) \to_{a.s.} \varphi_a(X_\infty), \quad \text{and,} \\ &Y_n^a \to_{a.s.} Y_\infty^a. \end{aligned}$$

Thus $Y_\infty^a = \varphi_a(X_\infty) = (X_\infty \vee a) + |a|$, and since Y_∞^a closes the sub-mg $\{Y_n^a, \mathcal{A}_n\}$, we have

$$E((X_\infty \vee a) + |a||\mathcal{A}_n) = E(Y_\infty^a|\mathcal{A}_n) \ge_{a.s.} Y_n^a = (X_n \vee a) + |a|.$$

This implies that

$$E(X_\infty \vee a|\mathcal{A}_n) \ge_{a.s.} X_n \wedge a.$$

Letting $a \searrow -\infty$ (through a countable sequence) yields, by application of the DCT for conditional expectations (PfS, Theorem 7.4.1 (19)),

$$E(X_\infty|\mathcal{A}_n) \ge_{a.s.} X_n.$$

Thus X_∞ (and the associated sigma-field $\mathcal{A}_\infty = \sigma[\cup_n \mathcal{A}_n]$) closes the sub-mg $\{X_n, \mathcal{A}_n\}$.

2. PfS, Exercise 13.3.6, page 359. Let $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ be a submg with with $X_n \geq 0$. Let $r > 1$. Then the X_n^r's are uniformly integrable if and only if the X_n^r process is integrable.

   Solution: Uniform integrability implies integrability, so it remains only to prove the reverse implication. Suppose that $\{X_n^r\}$ is integrable. Then $\{X_n\}$ is uniformly integrable, and hence by the s-martingale convergence theorem 13.3.1(B), $X_n \to X_\infty \in L_1$ where $\{X_n, \mathcal{A}_n\}_{n=0}^{\overline{\infty}}$ is a sub-mg; i.e. $E(X_\infty | \mathcal{A}_n) \geq X_n$ a.s. and

   $$E(X_\infty^r) = E(\liminf X_n^r) \leq \liminf E(X_n^r) \leq \sup_n E(X_n^r) < \infty$$

   by Fatou's lemma and integrability of $\{X_n^r\}$. Hence by the conditional Jensen inequality,

   $$E(X_n^r) \leq E\{E(X_\infty | \mathcal{A}_n)^r\} \leq E\{E(X_\infty^r | \mathcal{A}_n)\} = E(X_\infty^r)$$

   and it follows from Vitali's theorem that $\{X_n^r\}$ is uniformly integrable.

   Alternatively, by Doob's L_r−maximal inequality, since $\{X_n, \mathcal{A}_n\}$ is a sub-martingale,

   $$E\left\{\left(\max_{1\leq k\leq n} X_k\right)^r\right\} \leq \left(\frac{r}{r-1}\right)^r E|X_n|^r \,,$$

   and hence, by the monotone convergence theorem,

   $$E[\sup_{1\leq k<\infty} X_k^r] \leq \left(\frac{r}{r-1}\right)^r \sup_n E|X_n|^r < \infty \,.$$

   Thus with $Y \equiv \sup_{1\leq k<\infty} X_k$, it follows that

   $$\sup_n E\left\{X_n^r 1_{[X_n^r \geq \lambda]}\right\} \leq E(Y^r 1_{[Y^r \geq \lambda]} \to 0$$

   as $\lambda \to \infty$; i.e. $\{X_n^r\}$ is uniformly integrable.

3. PfS, Exercise 13.3.7, page 347. (i) Let $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ be a mg. Let $r > 1$. Then the following are equivalent:
   (10) The $|X_n|^r$−process is integrable.
   (11) $X_n \to_r X_\infty$.
   (12) The X_n's are uniformly integrable and $X_\infty \in L_r$.
   (13) The $|X_n|^r$'s are uniformly integrable.
   (14) $\{|X_n|^r, \mathcal{A}_n\}_{n=0}^{\bar{\infty}}$ is a submg and $E|X_n|^r \nearrow E|X_\infty|^r < \infty$.
   (*) $M^* \equiv \sup\{|X_n| : 0 \leq n \leq \infty\} \in L_r$ (via Doob's L_r-inequality).
   (ii) This theorem also holds for a submg when all $X_n \geq 0$.

Solution: Suppose that (10) holds. Then $|X_n|^r$ is an integrable sub-mg. Thus the $|X_n|^r$ are uniformly integrable by the preceding problem. Thus (13) holds. Suppose (13) holds. Then $\{X_n\}$ is uniformly integrable, and $X_n \to_{a.s.} X_\infty \in L_1$ and

$$E|X_\infty|^r = E(\liminf |X_n|^r) \le \liminf E|X_n|^r \le \sup_n E|X_n|^r < \infty\,,$$

so $X_\infty \in L_r$; i.e. (12) holds.
Suppose (12) holds. Then $\{|X_n|, \mathcal{A}_n\}_{n=0}^{\infty}$ is a sub-martingale by Theorem 13.3.1(B). Thus $|X_n| \le E(|X_\infty|\mathcal{A}_n)$, so $|X_n|^r \le \{E(|X_\infty||\mathcal{A}_n)\}^r \le E(|X_n|^r|\mathcal{A}_n)$ a.s., and hence $E|X_n|^r \le E|X_\infty|^r < \infty$; i.e. (10) holds.
Thus (10) iff (12) iff (13) holds.
Now (11) implies (10) since

$$E|X_n|^r \le c_r\{E|X_n - X_\infty|^r + E|X_\infty|^r\}$$

by the c_r-inequality.
Suppose that (13) holds. Then $X_n \to_{a.s.} X_\infty \in L_r$ (by (13) implies (12)), and since $\{|X_n|^r, \mathcal{A}_n\}_{n=0}^{\infty}$ is a sub-mg,

$$\limsup_{n\to\infty} E|X_n|^r \le E|X_\infty|^r < \infty\,.$$

Hence $X_n \to_r X_\infty$ by Vitali's theorem; i.e. (11) holds. Thus (10) iff (12) iff (13) iff (14).

4. Williams, PwM, Problem E10.1, page 231: (Polyá's urn). At time 0, an urn contains 1 black ball and 1 white ball. At each time $1, 2, 3, \ldots$, a ball is chosen at random from the urn and is replaced together with a new ball of the same color. Just after time n, there are therefore $n+2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. Let $M_n = (B_n+1)/(n+2)$, the proportion of black balls in the urn just after time n.
(i) Prove that (relative to a natural filtration $\mathcal{A}_n$ which you should specify), $\{M_n, \mathcal{A}_n\}$ is a martingale.
(ii) Prove that $P(B_n = k) = (n+1)^{-1}$ for $0 \le k \le n$. What is the distribution of $M \equiv \lim_n M_n$?
(iii) Prove that for $0 < \theta < 1$,

$$N_n^\theta \equiv \frac{(n+1)!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}$$

defines a martingale.

Proof. (i) Now

$$P(B_{n+1}+1 = k|B_n) = \begin{cases} \frac{B_n+1}{n+2}, & \text{if } k = B_n+2,\\ 1-\frac{B_n+1}{n+2}, & \text{if } k = B_n+1,\\ 0, & \text{otherwise.}\end{cases}$$

It follows that

$$\begin{aligned} E(B_{n+1}+1|B_n) &= (B_n+2)\frac{B_n+1}{n+2}+(B_n+1)\left(1-\frac{B_n+1}{n+2}\right) \\ &= \frac{(B_n+1+1)(B_n+1)-(B_n+1)^2}{n+2}+(B_n+1) \\ &= \frac{B_n+1}{n+2}+(B_n+1)=(B_n+1)\frac{n+3}{n+2}. \end{aligned}$$

Thus with $\mathcal{A}_n \equiv \sigma\{B_0, B_1, \ldots, B_n\}$, the sequence $M_n = (B_n+1)/(n+2)$ is adapted to $\mathcal{A}_n$ and $\{M_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is a martingale:

$$\begin{aligned} E(M_{n+1}|\mathcal{A}_n) &= E(M_{n+1}|B_n) = E\left(\frac{B_{n+1}+1}{n+3}|B_n\right) \\ &= \frac{E(B_{n+1}+1|B_n)}{n+3} = \frac{(B_n+1)\frac{n+3}{n+2}}{n+3} \\ &= \frac{B_n+1}{n+2} = M_n. \end{aligned}$$

(ii) From (i) it is immediate that the claim is true for $n=1$:

$$\begin{aligned} &P(B_1+1=k) = P(B_1+1=k|B_0) = \frac{1}{2} \quad \text{for} \quad k=1,2, \quad \text{so} \\ &P(B_1=k) = 1/2, \quad \text{for} \quad k=0,1. \end{aligned}$$

Now we proceed by induction: suppose the claim is true for $n \geq 1$; we want to show that it holds for $n+1$. Now for $k \in \{0, \ldots, n+1\}$

$$\begin{aligned} &P(B_{n+1}=k) \\ &\quad = EP(B_{n+1}+1=k+1|B_n) \\ &\quad = E\left\{\frac{B_n+1}{n+2}1\{B_n=k-1\}+\left(1-\frac{B_n+1}{n+2}\right)1\{B_n=k\}\right\} \\ &\quad = \frac{k}{n+2}P(B_n=k-1)+\left(1-\frac{k+1}{n+2}\right)P(B_n=k) \\ &\quad = \frac{k}{n+2}\frac{1}{n+1}+\left(1-\frac{k+1}{n+2}\right)\frac{1}{n+1}1\{k\leq n\} \\ &\quad = \begin{cases} \frac{1}{n+1}-\frac{1}{(n+1)(n+2)}=\frac{1}{n+1}\left(1-\frac{1}{n+2}\right)=\frac{1}{n+2}, & \text{if } k\leq n, \\ \frac{n+1}{n+2}\cdot\frac{1}{n+1}=\frac{1}{n+2}, & \text{if } k=n+1. \end{cases} \end{aligned}$$

Thus the claim holds for $n+1$. From (i) we have $EM_n = EM_0 = 1/2$ for all n. It follows from the s-mg convergence theorem that $M_n \rightarrow_{a.s.} M_\infty$ and in L_1. From the calculation above, $M_n \sim \text{Uniform}\{1/(n+2), \ldots, (n+1)/(n+2)\}$, and hence

$M_n \to_d$ Uniform$(0,1)$. Hence $M_\infty \sim$ Uniform$(0,1)$.
(iii) We compute directly using the basic result from part (i):

$$\begin{aligned}
E\left(N_{n+1}^{\theta} | B_n\right) &= E\left(\frac{(n+2)!}{B_{n+1}!(n+1-B_{n+1})!}\theta^{B_{n+1}}(1-\theta)^{n+1-B_{n+1}} \big| B_n\right) \\
&= (1-\theta)^{n+1} E\left(\frac{(n+2)!}{B_{n+1}!(n+1-B_{n+1})!}\left(\frac{\theta}{1-\theta}\right)^{B_{n+1}} \big| B_n\right) \\
&= (n+2)!(1-\theta)^{n+1}\left\{\frac{1}{(B_n+1)!(n-B_n)!}\left(\frac{\theta}{1-\theta}\right)^{B_n+1} \cdot \frac{B_n+1}{n+2}\right. \\
&\quad \left. + \frac{1}{B_n!(n+1-B_n)!}\left(\frac{\theta}{1-\theta}\right)^{B_n} \cdot \left(1-\frac{B_n+1}{n+2}\right)\right\} \\
&= (n+2)!(1-\theta)^{n+1}\left\{\frac{1}{(B_n)!(n-B_n)!}\left(\frac{\theta}{1-\theta}\right)^{B_n} \cdot \frac{\theta}{(1-\theta)(n+2)}\right. \\
&\quad \left. + \frac{1}{B_n!(n-B_n)!(n+1-B_n)}\left(\frac{\theta}{1-\theta}\right)^{B_n} \cdot \ \frac{n+1-B_n}{n+2}\right\} \\
&= (n+1)!(1-\theta)^n \frac{1}{(B_n)!(n-B_n)!}\left(\frac{\theta}{1-\theta}\right)^{B_n} \cdot (1-\theta) \cdot \left\{\frac{\theta}{1-\theta}+1\right\} \\
&= N_n^{\theta}.
\end{aligned}$$

Thus $\{N_n^{\theta}, \mathcal{A}_n\}_{n=0}^{\infty}$ is a martingale.