Statistics 523, Problem Set 3 Solutions

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1. Let $R \equiv \inf\{t > 1 : \ B_t = 0\}$ for Brownian motion B (started from $x \in \mathbb{R}$ at time $t = 0$), and let $L = \sup\{t \le 1 : \ B_t = 0\}$.
(i) Use the (ordinary) Markov property of Brownian motion to show that

$$P_x(R > 1 + t) = \int p_1(x, y) P_y(\tau_0 > t) dy, \qquad \text{and}$$
$$P_0(L \le t) = \int p_t(0, y) P_y(\tau_0 > 1 - t) dy$$

where $\tau_0 \equiv \inf\{t > 0 : B_t = 0\}$ and $p_t(x, y) = \phi((y - x)/\sqrt{t})/\sqrt{t}$ is the transition density for Brownian motion.
(ii) Use the distribution of τ_b we computed in class and in problem 4 of problem set 2 to show that

$$P_0(L \le s) = (2/\pi) \arcsin(\sqrt{s}),$$

and find the corresponding result for R.

Solution: (i) These both follow from the ordinary Markov property of Brownian motion: In the case of R we condition on $\mathcal{A}_1 = \sigma\{B_s : s \le 1\}$ to obtain

$$\begin{aligned} P_x(R > 1 + t) &= E_x P_x(R > 1 + t | \mathcal{A}_1) \\ &= E_x P_x(R > 1 + t | B_1) = \int_{-\infty}^{\infty} P(R > 1 + t | B_1 = y) p_1(x, y) dy \\ &= \int_{-\infty}^{\infty} P_y(\tau_0 > t) p_1(x, y) dy. \end{aligned} \tag{1}$$

In the case of L, conditioning on $\mathcal{A}_t$ yields

$$\begin{aligned} P_0(L \le t) &= E_0 P_0(L \le t | \mathcal{A}_t) = E_0 P_0(L \le t | B_t) \\ &= \int_{-\infty}^{\infty} P_y(\tau_0 > 1 - t) p_t(0, y) dy. \end{aligned} \tag{2}$$

(ii) From our calculations of the distribution of τ_b it follows that

$$P_y(\tau_0 > t) = \int_t^{\infty} \frac{|y|}{\sqrt{2\pi s^3}} \exp\left(-\frac{y^2}{2s}\right) ds.$$

Plugging this into (1) and taking $x = 0$ yields

$$\begin{aligned}
P_0(R > 1+t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \int_t^{\infty} \frac{|y|}{\sqrt{2\pi s^3}} \exp\left(-\frac{y^2}{2s}\right) ds \\
&= 2\int_0^{\infty}\int_{-\infty}^{\infty} 1_{[t\le s]} \frac{1}{\sqrt{2\pi}} \frac{|y|}{\sqrt{2\pi s^3}} \exp\left(-\frac{y^2}{2}\right) \exp\left(-\frac{y^2}{2s}\right) dyds \\
&= 2\int_0^{\infty} 1_{[t\le s]} \frac{1}{\sqrt{2\pi s^3}} \int_{-\infty}^{\infty} \frac{|y|}{\sqrt{2\pi}} \exp\left(-y^2\left(\frac{1}{2}+\frac{1}{2s}\right)\right) dyds \\
&= 2\int_0^{\infty} 1_{[t\le s]} \frac{\sqrt{\frac{s}{s+1}}}{\sqrt{2\pi s^3}} \int_{-\infty}^{\infty} \frac{|y|}{\sqrt{2\pi s/(s+1)}} \exp\left(-\frac{y^2}{2s/(s+1)}\right) dyds \\
&= 2\int_t^{\infty} \frac{\frac{s}{s+1}}{\sqrt{2\pi s^3}} E|Z| ds, \quad \text{where } Z \sim N(0,1) \\
&= 2\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\pi}} \int_t^{\infty} \frac{1}{\sqrt{s}(1+s)} ds.
\end{aligned}$$

Thus the random variable $S \equiv R - 1$ has density

$$f_S(s) = \frac{2}{\pi} \frac{1}{\sqrt{s}(1+s)} 1_{(0,\infty)}(s).$$

Similarly, in the case of L we have

$$P_y(\tau_0 > 1-t) = \int_{1-t}^{\infty} \frac{|y|}{\sqrt{2\pi s^3}} \exp(-\frac{y^2}{2s}) ds.$$

Plugging this into (2) yields

$$\begin{aligned}
P_0(L \le t) &= \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp(-y^2/2t) \int_{1-t}^{\infty} \frac{|y|}{\sqrt{2\pi s^3}} \exp(-\frac{y^2}{2s}) dsdy \\
&= 2\int_0^{\infty} \frac{1}{\sqrt{2\pi t}} \exp(-y^2/2t) \int_{1-t}^{\infty} \frac{y}{\sqrt{2\pi s^3}} \exp(-\frac{y^2}{2s}) dsdy \\
&= \frac{1}{\pi} \int_{1-t}^{\infty} (ts^3)^{-1/2} \int_0^{\infty} y \exp(-y^2(s+t)/2st) dyds \\
&= \frac{1}{\pi} \int_{1-t}^{\infty} (ts^3)^{-1/2} \frac{st}{s+t} ds \\
&= \frac{1}{\pi} \int_{1-t}^{\infty} \left(\frac{(t+s)^2}{st}\right)^{1/2} \frac{t}{(t+s)^2} ds \\
&= \frac{1}{\pi} \int_{1-t}^{\infty} \frac{1}{\sqrt{\frac{t}{t+s}\left(1-\frac{t}{t+s}\right)}} \frac{t}{(t+s)^2} ds \\
&= \frac{1}{\pi} \int_0^t \frac{1}{\sqrt{v(1-v)}} dv = \frac{2}{\pi} \arcsin\sqrt{t}.
\end{aligned}$$

by the change of variables $v = t/(t+s)$ so $dv = -tds/(t+s)^2$. Thus L has the arcsin distribution with density $\pi^{-1}(v(1-v))^{-1/2}1_{(0,1)}(v) = \text{Beta}(1/2, 1/2)$.

Note that the interval $[L, R]$ is an *excursion interval* for the Brownian motion B straddling the time $t = 1$. Chung (1976) finds the joint density of (L, R) (his $(\gamma(1), \beta(1))$):

$$f_{L,R}(u, v) = \frac{1}{2\pi\sqrt{u(v-u)^3}} 1\{0 < u < 1 < v\}.$$

It turns out that the process B_{ex} defined by

$$B_{ex}(t) = \frac{|B((1-t)L + tR)|}{\sqrt{R-L}}, \quad \text{for} \quad 0 \le t \le 1$$

is a *standard Brownian excursion.*

2. Let B denote standard Brownian motion starting from 0. Consider the stochastic process $\{\tau_b : b > 0\}$. Show that this process has stationary and independent increments. How is it related to the process $M_t \equiv \sup_{0 \le s \le t} B_s$?

   Solution: Let $a, b > 0$. Then by the strong Markov property of Brownian motion

$$\begin{aligned} P(\tau_{a+b} - \tau_a \le y, \tau_a \le x) &= EP(\tau_{a+b} - \tau_a \le y, \tau_a \le x | \mathcal{A}_{\tau_a}) \\ &= E\{P(\tau_b \le y) 1_{[\tau_a \le x]}\} = P(\tau_b \le y) P(\tau_a \le x). \end{aligned}$$

   Thus the increments of the process $b \mapsto \tau_b$ are independent and stationary. By looking at a graph of M_t, it becomes clear that the jumps in the process τ_b correspond to flat stretches in M_t. In fact, M_t and τ_b are just inverses of each other! It turns out that τ_b has a (completely asymmetric) stable law with index $\alpha = 1/2$ with Laplace transform $E\exp(-\lambda\tau_b) = \exp(-b\sqrt{2\lambda})$.

3. Exercise 12.7.3(a), PfS, page 325. (This is connected to Example 13.1.8, page 352.) That is, with $\tau_{ab} \equiv \tau_{-a} \wedge \tau_b$ with $a, b > 0$ we have

$$E\tau^2 \le 2\Gamma(2) 2^4 ab(a+b)^2 = 2^5 ab(a+b)^2.$$

   Solution: Now $Y_t(c) \equiv \exp(cB_t - c^2 t/2)$ is a martingale in t for each fixed $c \in \mathbb{R}$. Differentiating with respect to c and setting $c = 0$ yields

$$\begin{aligned} Y_t^{(1)}(c) &= Y_t(c)(B_t - ct) \overset{c=0}{=} B_t, \\ Y_t^{(2)}(c) &= Y_t(c)(B_t - ct)^2 - tY_t(c) \overset{c=0}{=\!=} B_t^2 - t, \\ Y_t^{(3)}(c) &= Y_t(c)(B_t - ct)^3 - Y_t(c)2(B_t - ct)t - tY_t(c)(B_t - ct) \\ &= Y_t(c)(B_t - ct)^3 - 3tY_t(c)(B_t - ct) \overset{c=0}{=} B_t^3 - 3tB_t, \\ Y_t^{(4)}(c) &= Y_t(c)(B_t - ct)^4 - 6tY_t(c)(B_t - ct)^2 + 3t^2 Y_t(c) \\ &\overset{c=0}{=} B_t^4 - 6tB_t^2 + 3t^2. \end{aligned}$$

Thus $\{B_t^4 - 6tB_t^2 + 3t^2 : t \geq 0\}$ is a martingale and for the stopping time τ we have

$$0 = EB_\tau^4 - 6E\{\tau B_\tau^2\} + 3E\tau^2,$$

or, equivalently,

$$3E\tau^2 = 6E\{\tau B_\tau^2\} - EB_\tau^4$$

where

$$EB_\tau^4 = (a^4 b + b^4 a)/(a+b) = ab(a^3+b^3)/(a+b) \leq ab(a+b)^2.$$

Furthermore, by the Cauch-Schwarz inequality

$$E\{\tau B_\tau^2\} \leq \{E\tau^2 \cdot EB_\tau^4\}^{1/2},$$

so we have, with $A^2 \equiv E\tau^2$ and $B^2 \equiv EB_\tau^4$,

$$3A^2 \leq 6AB - B^2 \qquad \text{or} \qquad 3A^2 - 6AB + B^2 \leq 0.$$

Thus $A \leq (6B + \sqrt{36B^2 - 4 \cdot 3B^2})/6 = B + B\sqrt{2/3} = B(1+\sqrt{2/3})$. We concude that

$$E(\tau^2) = A^2 \leq B^2(1+\sqrt{2/3})^2 \leq (1+\sqrt{2/3})^2 ab(a+b)^2.$$

4. Exercise 13.1.6, PfS, page 353.

Solution: In example 1.12, $N(t)$ is a Poisson process with intensity λ; and $M(t) = N(t) - \lambda t$ and $M^2(t) - \lambda t$ are both martingales. A natural exponential martingale to consider is

$$Y(t) \equiv Y_c(t) \equiv \frac{\exp(cM(t))}{E\exp(cM(t))}\,.$$

Since

$$\begin{aligned} E\exp(cM(t)) &= E\exp(c(N(t)-\lambda t)) \\ &= \exp(-c\lambda t)E\exp(c(N(t)) \\ &= \exp(-c\lambda t)\exp((e^c-1)\lambda t)\,, \end{aligned}$$

we find that

$$Y(t) = \exp(cN(t) - (e^c-1)\lambda t)\,.$$

I claim that $\{Y(t), \mathcal{A}_t\}_{t=0}^{\infty}$ is a martingale on $[0, \infty)$. To see this, note that for $0 \le s < t < \infty$ we have

$$\begin{aligned}
E\left(Y(t)|\mathcal{A}_s\right) &= E\left(\exp(cN(t) - (e^c - 1)\lambda t)|\mathcal{A}_s\right) \\
&= E\left(\exp(c(N(t) - N(s)) - (e^c - 1)\lambda(t - s))\exp(cN(s) - (e^c - 1)\lambda s)|\mathcal{A}_s\right) \\
&= Y(s)E\left(\exp(c(N(t) - N(s)) - (e^c - 1)\lambda(t - s))|\mathcal{A}_s\right) \qquad \text{a.s.} \\
&= Y(s)E\left(\exp(c(N(t) - N(s)) - (e^c - 1)\lambda(t - s))\right) \qquad \text{a.s.} \\
&\qquad\qquad \text{since } \; N(t) - N(s) \; \text{ is independent of } \; \mathcal{A}_s \\
&= Y(s) \cdot 1 = Y(s) \qquad \text{a.s.},
\end{aligned}$$

and hence $\{Y(t), \mathcal{A}_t\}_{t=0}^{\infty}$ is a martingale. Note that

$$\begin{aligned}
Y_c^{(1)}(t) \equiv \frac{d}{dc}Y_c(t)\Big|_{c=0} &= Y_c(t)\left(N(t) - e^c\lambda t\right)\Big|_{c=0} \\
&= Y_0(t)\left(N(t) - e^c\lambda t\right)\Big|_{c=0} \\
&= N(t) - \lambda t = M(t),
\end{aligned}$$

$$\begin{aligned}
Y_c^{(2)}(t) \equiv \frac{d^2}{dc^2}Y_c(t)\Big|_{c=0} &= Y_c(t)\left(N(t) - e^c\lambda t\right)^2\Big|_{c=0} + Y_c(t)\left(-e^c\lambda t\right)\Big|_{c=0} \\
&= M^2(t) - \lambda t,
\end{aligned}$$

$$\begin{aligned}
Y_c^{(3)}(t) \equiv \frac{d^3}{dc^3}Y_c(t)\Big|_{c=0} &= Y_c(t)\left(N(t) - e^c\lambda t\right)^3\Big|_{c=0} + Y_c(t)2\left(N(t) - e^c\lambda t\right)\left(-e^c\lambda t\right)\Big|_{c=0} \\
&\qquad + Y_c(t)\left(N(t) - e^c\lambda t\right)\left(-e^c\lambda t\right)\Big|_{c=0} - Y_c(t)e^c\lambda t \\
&= Y_c(t)\left\{(N(t) - e^c\lambda t)^3 - 3e^c\lambda t(N(t) - e^c\lambda t) - e^c\lambda t\right\}\Big|_{c=0} \\
&= M^3(t) - 3\lambda t M(t) - \lambda t,
\end{aligned}$$

and

$$\begin{aligned}
Y_c^{(4)}(t) &\equiv \frac{d^4}{dc^4}Y_c(t)\Big|_{c=0} \\
&= Y_c(t)(N(t) - e^c\lambda t)\left\{(N(t) - e^c\lambda t)^3 - 3e^c\lambda t(N(t) - e^c\lambda t) - e^c\lambda t\right\} \\
&\qquad + Y_c(t)\left\{3(N(t) - e^c\lambda t)^2(-e^c\lambda t) - 3e^c\lambda t(N(t) - e^c\lambda t)\right. \\
&\qquad \left. - 3e^c\lambda t(N(t) - e^c\lambda t)(-e^c\lambda t) - e^c\lambda t\right\}\Big|_{c=0} \\
&= Y_c(t)\left\{(N_t - e^c\lambda t)^4 - 6e^c\lambda t(N_t - e^c\lambda t)^2\right. \\
&\qquad \left. -4e^c\lambda t(N_t - e^c\lambda t) + 3(e^c\lambda t)^2 - e^c\lambda t\right\}\Big|_{c=0} \\
&= M_t^4 - 6\lambda t M_t^2 - 4\lambda t M_t + 3(\lambda t)^2 - \lambda t.
\end{aligned}$$

Note that the expected value of the right side is

$$\lambda t + 3(\lambda t)^2 - 6(\lambda t)^2 - 4 \cdot 0 + 3(\lambda t)^2 - \lambda t = 0.$$