Statistics 523, Problem Set 2 Solutions

Wellner; 4/18/10

1. PfS, Exercise 12.9.3, page 332; i.e. prove that $V_n(1)$ as defined in (12.9.1) converges a.s. to ∞ when the partition $\mathcal{P}_n$ is given by $t_{n,k} = k/2^n$.

   Solution: For the partition $\{t_{n,k}\} = \{k/2^n\}_{k=0}^{2^n}$, we see that since the random variables $\{B(k/2^n) - B((k-1)/2^n) : 1 \le k \le 2^n\}$ are i.i.d. with $B(k/2^n) - B((k-1)/2^n) \stackrel{d}{=} 2^{-n/2} Z$ with $Z \sim N(0,1)$ it follows that

   $$EV_n(1) = 2^{n/2} E|Z|, \qquad \text{and} \qquad Var(V_n(1)) = 1,$$

   so that

   $$EV_n^2(1) = Var(V_n(1)) + (EV_n(1))^2 = 1 + 2^n (E|Z|)^2.$$

   Thus by the Paley-Zygmund inequality, for any $\delta > 0$ it follows that

   $$P(V_n(1) \ge \delta EV_n(1)) \ge (1-\delta)^2 \frac{(EV_n(1))^2}{EV_n^2(1)} = (1-\delta)^2 \frac{(2^{n/2}\sqrt{2/\pi})^2}{1 + 2^n(2/\pi)} \to (1-\delta)^2.$$

   But $V_n(1) \le V_{n+1}(1)$ by the triangle inequality, so $V_n \nearrow$ and since $EV_n(1) \to \infty$ as $n \to \infty$, $\delta EV_n(1) \ge M$ for each fixed (large) M. Thus it follows that

   $$\begin{aligned} P(\lim_{n\to\infty} V_n \ge M) &\ge \lim_{n\to\infty} P(V_n \ge M) \\ &\ge \liminf_{n\to\infty} P(V_n \ge \delta EV_n(1)) \ge (1-\delta)^2 \end{aligned}$$

   for each fixed δ and M. Letting $M \to \infty$ and $\delta \to 0$ yields the conclusion.

2. Prove Theorem 12.9.1 (c): i.e. prove that $V_n(2) \to_2 1$ if $\|\mathcal{P}_n\| \to 0$.

   Solution: Now the random variables $\{B(t_{n,k}) - B(t_{n,k-1}) : \ 1 \le k \le n\}$ are independent and

   $$B(t_{n,k}) - B(t_{n,k-1}) \sim N(0, t_{n,k} - t_{n,k-1}) \stackrel{d}{=} (t_{n,k} - t_{n,k-1})^{1/2} Z.$$

   Thus

   $$EV_n(2) = \sum_{k=1}^n E(B(t_{n,k}) - B(t_{n,k-1}))^2 = \sum_{k=1}^n (t_{n,k} - t_{n,k-1}) = 1$$

while

$$\begin{aligned} Var(V_n(2)) &= \sum_{k=1}^{n} (t_{n,k} - t_{n,k-1})^2 Var(B_1^2) \\ &\leq 2\|\mathcal{P}_n\| \sum_{k=1}^{n} |t_{n,k} - t_{n,k-1}| = 2\|\mathcal{P}_n\| \to 0 \end{aligned}$$

since $\|\mathcal{P}_n\| \to 0$. But $Var(V_n(2)) = E(V_n(2) - EV_n(2))^2 = E(V_n(2) - 1)^2$, so $V_n(2) \to_2 1$.

3. For a stopping time τ and a Brownian motion B we define the shift operator θ_τ by $\theta_\tau B = B(\tau + t)$ on $[\tau < \infty]$ and 0 otherwise. Suppose that $Y : [0, \infty) \times C[0, \infty) \to \mathbb{R}$ is bounded and $\mathcal{B} \times \mathcal{C}_{[0,\infty)}-$ measurable. If $\{B_t : t \geq 0\}$ is standard Brownian motion on $[0, \infty)$, then we can consider the new process $Y_t \equiv Y(t, B)$ and $Y_\tau \circ \theta_\tau \equiv Y(\tau, \theta_\tau B) = Y(\tau, B(\tau + \cdot))$.

   Show that the strong Markov property can be reformulated as follows: for all $x \in \mathbb{R}$ and Y as stated,

$$E_x(Y_\tau \circ \theta_\tau | \mathcal{A}_\tau) = E_{B(\tau)} Y_\tau \quad \text{on} \quad [\tau < \infty].$$

Solution: The claimed equality means that we want to show: for $A \in \mathcal{A}_\tau$,

$$E_x(1_A 1_{[\tau<\infty]} Y_\tau \circ \theta_\tau) = E\{1_A 1_{[\tau<\infty]} E_x E_{B(\tau)}(Y_\tau)\}.$$

But we compute as follows:

$$\begin{aligned} E_x(1_A 1_{[\tau<\infty]} Y_\tau \circ \theta_\tau) &= E_x(Y(\tau, B(\tau + \cdot)) 1_A 1_{[\tau<\infty]}) \\ &= E_0(Y(\tau, B(\tau + \cdot) + x - (B(\tau) + x) + B(\tau)) 1_{A \cap [\tau<\infty]}) \\ &= E_0(Y(\tau, \tilde{B}(\cdot) + B(\tau)) 1_{A \cap [\tau<\infty]}) \\ &\qquad \text{by Theorem 12.5.1 with } \tilde{B} \text{ independent of } \mathcal{A}_\tau \\ &= E_0\{1_{A \cap [\tau<\infty]} E(Y(\tau, \tilde{B}(\cdot) + B(\tau)) | \mathcal{A}_\tau)\} \\ &\qquad \text{since } A \cap [\tau < \infty] \in \mathcal{A}_\tau \\ &= E\{1_{A \cap [\tau<\infty]} E_{B(\tau)}(Y_\tau)\} \end{aligned}$$

where the last line holds since $B(\tau)$ is $\mathcal{A}_\tau-$measurable and $\tilde{B}$ is independent of $\mathcal{A}_\tau$.

4. Let $\tau \equiv \tau_b \equiv \inf\{t : \mathbb{B}(t) \ge b\}$ be the first hitting time of $b > 0$ for a Brownian motion b. Find the distribution function and density of τ.

Solution: We found the distribution function $F(t) = P(\tau \le t)$ in class, since we computed $F(t) = P(\tau \le t) = 2P(\mathbb{B}(t) \ge b) = 2(1 - \Phi(b/\sqrt{t}))$. Thus the density function f of τ is given by

$$\begin{aligned} f(t) &= -2\phi(b/\sqrt{t})b(-1/2)t^{-3/2} = \frac{b}{t^{3/2}}\phi(b/\sqrt{t}) \\ &= \frac{b}{\sqrt{2\pi t^3}} \exp\left(-\frac{b^2}{2t}\right) 1_{(0,\infty)}(t). \end{aligned}$$

Note that $f(t) \sim b/(\sqrt{2\pi t^3})$ as $t \to \infty$, in the sense that

$$\frac{f(t)}{b/\sqrt{2\pi t^3}} \to 1 \qquad \text{as } \; t \to \infty.$$

It follows that $E\tau^{1/2} = \infty$ while $E\tau^r < \infty$ if $r < 1/2$.