Statistics 523, Problem Set 7, Corrected Version

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Reading: Shorack, PfS; Chapter 12, pages 319-321; 326-348.
Shorack, PfS; Chapter 13, pages 352-353; 383-389.

Due: Friday, May 21, 2010.

1. PfS, Exercise 13.1.4, page 353.

2. PfS, Exercise 13.1.5, page 353, (corrected slightly): Verify the claims made in example 1.11: Suppose that the rv's $\xi_1, \xi_2, \ldots$ are i.i.d. Uniform$(0,1)$, and let $\mathbb{N}_n(t) \equiv n\mathbb{G}_n(t) \equiv$ (the number of ξ_i's $\le t$). Show that

$$\begin{aligned} \mathbb{M}_n(t) &\equiv \mathbb{N}_n(t) - \int_0^t \frac{n(1-\mathbb{G}_n(r-))}{1-r} dr \\ &= \sqrt{n}\left\{ \mathbb{U}_n(t) + \int_0^t \frac{\mathbb{U}_n(r-)}{1-r} dr \right\} \end{aligned}$$

is a martingale with covariance $n(s \wedge t)$.

3. PfS, Exercise 12.8.1, page 328: Let $X_0 \equiv 0$, and $X_1, \ldots$ be i.i.d. $(0, \sigma^2)$. Define $S_k \equiv X_1 + \cdots + X_k$ for each integer $k \ge 0$.
(a) Find the asymptotic distribution of $(S_1 + S_2 + \cdots + S_n)/c_n$ for an appropriate c_n.
(b) Determine a representation for the asymptotic distribution of the "absolute area" under the partial sum process, as given by $(|S_1| + \cdots + |S_n|)/c_n$.

4. (a) PfS, Exercise 12.10.1, page 338. Here is a rephrasing: suppose that $Y_1, \ldots, Y_{n+1}$ are i.i.d. exponential(1) random variables, and set $T_k = Y_1 + \cdots + Y_k$ for $k = 1, 2, \ldots, n+1$. Suppose that $\xi_1, \xi_2, \ldots$ are i.i.d. Uniform$(0,1)$ and let $0 \le \xi_{n:1} \le \xi_{n:1} \le \ldots \le \xi_{n:n} \le 1$ be the ordered statistics of $\xi_1, \ldots, \xi_n$. Show that

$$\underline{U}_n \equiv \left(\frac{T_1}{T_{n+1}}, \ldots, \frac{T_n}{T_{n+1}} \right) \stackrel{d}{=} (\xi_{n:1}, \ldots, \xi_{n:n}) \equiv \underline{\xi}_{(n)}.$$

(b) Show that the construction in (a) for a fixed n is not correct jointly in n: i.e. show that

$$(\underline{U}_n, \underline{U}_{n+1}) \overset{d}{\neq} (\underline{\xi}_{(n)}, \underline{\xi}_{(n+1)}).$$

Hint: consider $n = 1$.

5. **Bonus Problem 1.** Suppose that $\mathbb{B}$ is standard Brownian motion on $(C[0,\infty), \mathcal{C}_{[0,\infty)})$, and let its distribution be denoted by $P = P_0$. Let $\mathbb{B}_\mu(t) \equiv \mathbb{B}(t) + \mu t$ be Brownian motion with drift μ, and let P_μ denote the distribution of $\mathbb{B}_\mu$ on $(C[0,\infty), \mathcal{C}_{[0,\infty)})$. Set $Y(t) \equiv \exp(\mu\mathbb{B}(t) - \mu^2 t/2)$. For $t > 0$ let $P_{0,t}$ and $P_{\mu,t}$ denote the distributions P_0 and P_μ restricted to $\mathcal{A}_t \equiv \{\mathbb{B}(s) : s \le t\}$. Show that the Radon - Nikodym derivative $dP_{\mu,t}/dP_{0,t} = Y(t)$.

6. **Bonus Problem 2.** Suppose that $\mathbb{B}_\mu$ is Brownian motion with drift $\mu > 0$ as in problem 2, and let $\tau \equiv \inf\{t > 0 : \mathbb{B}_\mu(t) = a\}$, $a > 0$. Use the result of bonus problem 1 together with results from problem set #2 (problem 2.4) to find the distribution of τ. You should find that

$$\begin{aligned} P_\mu(\tau > t) &= P_\mu(\mathbb{B}_\mu(s) < a,\ 0 \le s \le t) \\ &= \Phi\left(\frac{a - \mu t}{\sqrt{t}}\right) - e^{2\mu}\Phi\left(\frac{-a - \mu t}{\sqrt{t}}\right) \end{aligned}$$

and

$$f_\tau(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{(a - \mu t)^2}{2t}\right) \qquad \text{for } t \ge 0\,.$$

This is the *inverse Gaussian* density. [Note that this reduces to the density of τ_a from problem 2.4 when $\mu = 0$.