Statistics 523, Problem Set 5

Wellner; April 30, 2010

Reading: Shorack, PfS; Chapter 12, pages 247 - 283.

Due: Friday, May 7, 2010.

Reminder: Due on May 3: an outline (about one page) of your project paper, with references.

1. Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with values in the measurable space $(S, \mathcal{S})$ be defined on $(\Omega, \mathcal{A}, P)$. Let $\mathcal{A}_n$ be the $\sigma-$field generated by all functions of $(X_1, X_2, \ldots)$ that are symmetric in their first n arguments. Prove that a sequence $\{U_n\}$ of $U-$statistics based on a fixed symmetric kernel h of order r (so that $h : S^r \to \mathbb{R}$) is a reverse martingale for $n \ge r$ with respect to the filtration $\mathcal{A}_r \supset \mathcal{A}_{r+1} \supset \cdots$.

2. Suppose that the kernel h in problem 1 satisfies $E|h(X_1, \ldots, X_r)| < \infty$. Show that the corresponding sequence of U statistics $\{U_n\}_{n \ge r}$ satisfies $U_n \to_{a.s.,1} Eh(X_1, \ldots, X_r)$. (For $r > 1$ the given condition is not necessary, but a simple necessary and sufficient condition appears to be unknown.) Use your result for $\{U_n\}$ to show a corresponding result for the sequence of $V-$statistics $\{V_n\}$ associated with $\{U_n\}$ under reasonable additional hypotheses.

3. Williams, PwM, Problem E10.8, page 234.

4. (a) Suppose that $X_1, X_2, \ldots$ are independent with probability distributions $(P_1, P_2, \ldots)$ where $P_k = N(0, 1)$ for each k or independent with distributions $(Q_1, Q_2, \ldots)$ where $Q_k = N(c_k, 1)$ for some sequence of numbers c_k with $c_k \to 0$. Calculate $H(P_k, Q_k)$ in terms of c_k where $H(P, Q)$ is the Hellinger distance between P, Q given by

$$H^2(P, Q) = \frac{1}{2} \int \{\sqrt{p} - \sqrt{q}\}^2 d\mu$$

where $p = dP/d\mu$, $q = dQ/d\mu$ for some dominating measure μ (recall that $\mu = P + Q$ always works). If $Y_k \equiv dQ_k/dP_k$ for $k = 1, 2, \ldots$, and

we define $M_n \equiv \prod_{k=1}^n Y_k$, for what sequences c_k does $M_n \rightarrow_{a.s.} 1$ under $P^{infty} = \prod_{k=1}^{\infty} P_k$? For what sequences does $M_n \rightarrow_{a.s.} 0$ under P^{∞}?
(b) Consider a similar example when $P_k = \exp(1)$ for all k and $Q_k = \exp(c_k)$ with $c_k \rightarrow 1$.
Hint: compute in terms of the Hellinger affinity $\rho(P_k, Q_k) \equiv \int \sqrt{p_k q_k} d\mu$.