

## Statistics 523, Problem Set 2

Wellner; 4/9/10

**Reading:** Shorack, PFS; Chapter 12, section 7, pages 322-325;  
Chapter 13, pages 349-382.

**Due:** Friday, April 16, 2010.

1. PFS, Exercise 12.9.3, page 332; i.e. prove that  $V_n(1)$  as defined in (12.9.1) converges a.s. to  $\infty$  when the partition  $\mathcal{P}_n$  is given by  $t_{n,k} = k/2^n$ .
2. Prove Theorem 12.9.1 (c): i.e. prove that  $V_n(2) \rightarrow_2 1$  if  $\|\mathcal{P}_n\| \rightarrow 0$ .
3. For a stopping time  $\tau$  and a Brownian motion  $B$  we define the shift operator  $\theta_\tau$  by  $\theta_\tau B = B(\tau + t)$  on  $[\tau < \infty]$  and 0 otherwise. Suppose that  $Y : [0, \infty) \times C[0, \infty) \rightarrow \mathbb{R}$  is bounded and  $\mathcal{B} \times \mathcal{C}_{[0, \infty)}$ -measurable. If  $\{B_t : t \geq 0\}$  is standard Brownian motion on  $[0, \infty)$ , then we can consider the new process  $Y_t \equiv Y(t, B)$  and  $Y_\tau \circ \theta_\tau \equiv Y(\tau, \theta_\tau B) = Y(\tau, B(\tau + \cdot))$ .

Show that the strong Markov property can be reformulated as follows: for all  $x \in \mathbb{R}$  and  $Y$  as stated,

$$E_x(Y_\tau \circ \theta_\tau | \mathcal{A}_\tau) = E_{B(\tau)} Y_\tau \quad \text{on } [\tau < \infty].$$

4. Let  $\tau \equiv \tau_b \equiv \inf\{t : \mathbb{B}(t) \geq b\}$  be the first hitting time of  $b > 0$  for a Brownian motion  $b$ . Find the distribution function and density of  $\tau$ .
5. Tell me that title or topic for your project (paper and talk).
6. **Optional bonus problem:** Let  $\{\mathbb{B}_t\}$  be standard Brownian motion. Let  $\tau(\omega)$  be the smallest  $t$  at which the sample path  $\mathbb{B}(\cdot, \omega)$  achieves its maximum value  $M(\omega) \equiv \sup_{0 \leq t \leq 1} \mathbb{B}(t, \omega)$  on  $[0, 1]$ . (i) Show that  $\tau$  is not a stopping time with respect to the natural filtration  $\mathcal{A}_t \equiv \sigma\{\mathbb{B}_s : s \leq t\}$ .  
(ii) Show that  $\tau$  is  $\mathcal{A}_1$ -measurable.