Statistics 522, Problem Set 8 Solutions

Wellner; 3/5/99

1. Exercise 11.7.2, PfS page 241: If $F_n \to_d F$ for a continuous df F, then $\|F_n - F\| \to 0$.

   Solution: Let M be a (large) positive integer, and set $x_{M,j} \equiv x_j = F^{-1}(j/(M+1))$ for $j = 1, \ldots, M$, and let $x_{M,0} \equiv -\infty$, $x_{M,M+1} \equiv \infty$. Then for $x \in [x_{j-1}, x_j]$ we have

$$F_n(x) - F(x) \begin{cases} \le F_n(x_j) - F(x_{j-1}) \le F_n(x_j) - F(x_j) + 1/M \\ \ge F_n(x_{j-1}) - F(x_j) \ge F_n(x_{j-1}) - F(x_{j-1}) - 1/M \end{cases},$$

   and hence

$$\begin{aligned} \|F_n - F\| &\le \max_{1\le j\le M+1} \sup_{x_{j-1}\le x\le x_j} |F_n(x) - F(x)| \\ &\le \max_{1\le j\le M} |F_n(x_j) - F(x_j)| + 1/M \\ &\to 0 + 1/M \end{aligned}$$

   since $F_n(x) \to F(x)$ at all x in view of F being continuous. But since M is arbitrary, this can be made arbitrarily small; i.e. $\|F_n - F\| \to 0$.

2. Exercise 11.7.3, PfS page 241: Suppose that $X_n \sim F_n$. Show that $\{F_n : n \ge 1\}$ is tight if either:
   (a) $\limsup_n E|X_n|^r < \infty$ for some $r > 0$. (b) $F_n \to_d F$.

   Solution: (a) Let $\epsilon > 0$. Now by Markov's inequality

$$\limsup_{n\to\infty} P(|X_n| > M) \le \frac{\limsup_{n\to\infty} E|X_n|^r}{M^r} < \epsilon$$

   for $M > M(r,\epsilon) \equiv (\limsup_{n\to\infty} E|X_n|^r/\epsilon)^{1/r}$. Thus there is an $N \equiv N(\epsilon, r)$ such that $\sup_{n>N} P(|X_n| > M) \le 2\epsilon$. But since $F_1, \ldots, F_N$ are df's, there exists a $K = K_\epsilon$ so large that $\max_{1\le n\le N} P(|X_n| > K) < 2\epsilon$. Taking $R = R_\epsilon = \max\{M, K\}$ we have

$$\sup_{1\le n<\infty} P(|X_n| > R) < 2\epsilon\,.$$

Thus $\{F_n\}$, the family of distributions of $\{X_n\}$, is tight.
(b) Let $\epsilon > 0$. Let $r = r(\epsilon) \in C_F$ be so large that $1 - F(r) < \epsilon/4$, and let $l = l(\epsilon) \in C_F$ be so small that $F(l) < \epsilon/4$. Now there exists an $N = N_\epsilon$ so large that

$$|F_n(r) - F(r)| < \epsilon/4 \qquad \text{for all} \qquad n > N$$

and

$$|F_n(l) - F(l)| < \epsilon/4 \qquad \text{for all} \qquad n > N\,.$$

Then we have

$$\begin{aligned}\sup_{n>N} F_n([l,r]^c) &\leq F(l) + (1 - F(r)) \\ &\qquad + |F_n(l) - F(l)| + |F_n(r) - F(r)| \\ &\leq \epsilon/4 + \epsilon/4 + \epsilon/4 + \epsilon/4 = \epsilon\end{aligned}$$

But since $F_1, \ldots, F_N$ are distribution functions, we can easily find an interval $[l', r']$, such that

$$\max_{1\leq n\leq N} F_n([l', r']^c) < \epsilon\,,$$

and hence the interval $[a, b] \equiv [l \wedge l', r \vee r']$ satisfies

$$\sup_{1\leq n<\infty} F_n([a,b]^c) < \epsilon\,;$$

i.e $\{F_n\}$ is tight.

3. Exercise 11.8.3, PfS page 241: Suppose that $\log X \sim N(0,1)$; thus the density of X with respect to Lebesgue measure is

$$f_X(x) = x^{-1}\exp(-(\log x)^2/2)/\sqrt{2\pi}, \qquad x > 0\,.$$

For $|a| \leq 1$, let Y have the density function

$$f_Y(y) = f_Y(y;a) = f_X(y)[1 + a\sin(2\pi\log y)], \qquad y > 0\,.$$

Show that X and Y have exactly the same moments.

Solution: It is clear that Y has the same moments as X if and only if

$$\int_0^\infty y^k f_X(y)\sin(2\pi\log y)dy = 0$$

for $k = 1, 2, \ldots$. Changing variables, it follows that

$$\begin{aligned}\int_0^\infty y^k f_X(y)\sin(2\pi\log y)dy &= \frac{1}{\sqrt{2\pi}}\int_0^\infty y^{k-1}\exp(-(\log y)^2/2)\sin(2\pi\log y)dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{kx}e^{-x^2/2}\sin(2\pi x)dx\,.\end{aligned}$$

Now the function in the integrand, $g(x) = e^{kx}e^{-x^2/2}\sin(2\pi x)$, is an entire function in the complex plane, and it follows from Cauchy's formula that the integral $\int_\Gamma g(z)dz = 0$ for any closed path Γ in the complex plane. In particular, taking the closed path to consist of the x axis from $-R$ to R, followed by the semicircle of radius R in the lower half plane connecting $(+R, 0)$ to $(-R, 0)$, then $\int_\Gamma g(z)dz = 0$. Now for $z = -Re^{i\theta}$, $\theta \in [0, \pi]$,

$$\begin{aligned}g(z) &= \exp(-kRe^{i\theta})\exp(-R^2/2)\exp(i2\pi Re^{i\theta}) \\ &= \exp(-kR(\cos\theta + i\sin\theta))\exp(-R^2/2)\exp(i2\pi R(\cos\theta + i\sin\theta)) \\ &= \exp(-kR(\cos\theta + i\sin\theta))\exp(-R^2/2)\exp(-2\pi R\sin\theta)\exp(i2\pi R\cos\theta)\end{aligned}$$

so that

$$|g(z)| \le \exp(-R(k\cos\theta + 2\pi\sin\theta))\exp(-R^2/2) \to 0$$

as $R \to \infty$. Thus taking the limit on R across the identity $\int_{\Gamma(R)} g(z)dz = 0$ yields

$$\int_{-\infty}^\infty e^{kx}e^{-x^2/2}\sin(2\pi x)dx = 0\,.$$

4. Exercise 11.8.5, PfS page 241. Show that the $N(0,1)$ distribution is uniquely determined by its moments (by appealing to the previous proposition).

   Solution: The moments of the $N(0,1)$ distribution are $E(Z^k) = 0$ for k odd, and $E(Z^k) = \int z^k\phi(z)dz = (k-1)(k-3)\cdots 3\cdot 1$. k even.

Thus $\mu_{2k} = (2k-1)(2k-3)\cdots 3\cdot 1$ where there are k factors. Thus $\mu_{2k} \le (2k-1)^k$, and $\mu_{2k}^{1/2k} \le (2k-1)^{1/2}$, so that

$$\limsup_{k\to\infty} \frac{1}{k}\mu_{2k}^{1/2k} \le \limsup_{k\to\infty} \frac{(2k-1)^{1/2}}{k} .$$

Thus by part (c) of exercise 11.8.4, the $N(0,1)$ df is the unique d.f. with the moments μ_k.

5. Exercise 11.1.1, PfS page 222: Let rv's X and Δ be arbitrary. Let $\epsilon > 0$. Suppose that G has derivative g. Then

$$\|F_{X+\Delta} - G\| \le \|F_X - G\| + \epsilon\|g\| + P(|\Delta| > \epsilon) .$$

Solution: First write

$$\begin{aligned} F_{X+\Delta}(x) - G(x) &= \int_{(-\infty,x]} d(F_{X+\Delta} - G) \\ &= P(X+\Delta \le x) - G(x) \\ &= P(X+\Delta \le x, |\Delta| \le \epsilon) + P(X+\Delta \le x, |\Delta| > \epsilon) - G(x) \\ &\quad \begin{cases} \le & P(X \le x+\epsilon) - G(x) + P(|\Delta| > \epsilon) \\ \ge & P(X \le x-\epsilon) - G(x) \end{cases} . \end{aligned}$$

It follows that

$$\begin{aligned} |F_{X+\Delta}(x) - G(x)| \le\ & \max\{|F_X(x+\epsilon) - G(x+\epsilon)|, |F_X(x-\epsilon) - G(x-\epsilon)|\} \\ &+ \max\{|G(x+\epsilon) - G(x)|, |G(x-\epsilon) - G(x)|\} \\ &+ P(|\Delta| > \epsilon) . \end{aligned}$$

From this it follows easily that the asserted inequality holds.