Statistics 522, Problem Set 7 Solutions

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1. Exercise 18.4.1, PfS page 409: If $Z_{n+1} = \sum_{j=1}^{Z_n} X_{nj}$, where X_{nj} are i.i.d. as X with $E(X) = m$, and $W_n \equiv Z_n/m^n$, then

$$Var(W_n) = \left\{ \begin{array}{ll} n\sigma^2 & \text{if } m = 1 \\ \sigma^2 \frac{1-m^{-n}}{m(m-1)} & \text{if } m \neq 1 \end{array} \right\} .$$

To show this, note that

$$Var(W_n) = EVar(W_n|Z_{n-1}) + Var(E(W_n|Z_{n-1})) = E\left\{\frac{Z_{n-1}\sigma^2}{m^{2n}}\right\} + Var(\frac{Z_{n-1}m}{m^n}) .$$

When $m = 1$, this yields

$$Var(W_n) = \sigma^2 E(Z_{n-1}) + Var(W_{n-1}) = \sigma^2 + Var(W_{n-1}) = \ldots = n\sigma^2$$

by recursion. When $m \neq 1$, we find that

$$Var(W_n) - Var(W_{n-1}) = \sigma^2 \frac{m^{n-1}}{m^{2n}} = \sigma^2 \frac{1}{m^{n+1}} ,$$

and therefore

$$\begin{aligned} Var(W_n) &= \sigma^2 \sum_{k=1}^{n} \frac{1}{m^{k+1}} \\ &= \sigma^2 \left\{ \sum_{k=1}^{\infty} \frac{1}{m^{k+1}} - \sum_{k=n+1}^{\infty} \frac{1}{m^{k+1}} \right\} \\ &= \sigma^2 \left\{ \frac{1}{m^2} \frac{1}{1-1/m} - \frac{1}{m^{n+2}} \frac{1}{1-1/m} \right\} \\ &= \sigma^2 \frac{1}{m(m-1)} \left\{ 1 - \frac{1}{m^n} \right\} . \end{aligned}$$

To see that $f_{n+1}(s) = f_n(f(s))$, write

$$
\begin{aligned}
f_{n+1}(s) &= E(s^{Z_{n+1}}) = E\left(s^{\sum_{j=1}^{Z_n} X_{nj}}\right) \\
&= E\left\{E\left(s^{\sum_{j=1}^{Z_n} X_{nj}} | Z_n\right)\right\} \\
&= E\left\{E\left(\prod_{j=1}^{Z_n} s^{X_{nj}} | Z_n\right)\right\} \\
&= E\left\{\prod_{j=1}^{Z_n} E\left(s^{X_{nj}} | Z_n\right)\right\} \\
&= E\left\{\prod_{j=1}^{Z_n} E\left(s^{X_{nj}}\right)\right\} \\
&= E\left\{\prod_{j=1}^{Z_n} f(s)\right\} \\
&= E\left\{f(s)^{Z_n}\right\} = f_n(f(s)) .
\end{aligned}
$$

Note that this implies that $f_n(s) = f_{n-1}(f(s))$, for each $n = 2, 3, \ldots$, and hence

$$
f_{n+1}(s) = f_n(f(s)) = f_{n-1}(f(f(s))) = f_{n-1}(f_2(s)) = \ldots = f(f_n(s)) .
$$

2. Exercise 18.4.2, PfS page 410. Suppose that $X_1, X_2, \ldots$ are i.i.d. F, that $h : R^2 \to R$ is symmetric, and $E|h(X_1, X_2)| < \infty$. Set $\mathcal{A}_n \equiv \sigma[\underline{X}_{n:n}, X_{n+1}, X_{n+2}, \ldots]$, and

$$
U_n = \binom{n}{2}^{-1} \sum_{1 \le i < j \le n} h(X_i, X_j) .
$$

Then

$$
\begin{aligned}
E[h(X_1, X_2)|\mathcal{A}_n] &= E[h(X_1, X_2)|\underline{X}_{n:n}, X_{n+1}, X_{n+2}, \ldots] \\
&= E[h(X_1, X_2)|\underline{X}_{n:n}] \\
&= \binom{n}{2}^{-1} \sum_{1 \le i < j \le n} E[h(X_i, X_j)|\underline{X}_{n:n}]
\end{aligned}
$$

$$
\begin{aligned}
&= E[U_n|\underline{X}_{n:n}] \\
&\quad \text{by linearity of conditional expectation} \\
&\quad \text{and by definition of } U_n \\
&= U_n
\end{aligned}
$$

since

$$2\binom{n}{2}U_n = \sum_{1\le i\ne j\le n} h(X_i,X_j) = \int\int h(x,y)d\mathbb{F}_n(x)d\mathbb{F}_n(y) - \int h(x,x)d\mathbb{F}_n(x)$$

(1)
where

$$\mathbb{F}_n(x) = n^{-1}\sum_{i=1}^n 1_{[X_i\le x]} = n^{-1}\sum_{i=1}^n 1_{[X_{n:i}\le x]}$$

so that U_n is measurable with respect to $\sigma[\underline{X}_{n:n}]$. Thus

$$U_n = E[h(X_1,X_2)|\mathcal{A}_n]\,.$$

Since $\mathcal{A}_n \supset \mathcal{A}_{n+1} \supset \ldots$, it follows that

$$E(U_n|\mathcal{A}_{n+1}) = E\left\{E[h(X_1,X_2)|\mathcal{A}_n]|\mathcal{A}_{n+1}\right\} = E[h(X_1,X_2)|\mathcal{A}_{n+1}] = U_{n+1} \qquad \text{a.s.}.$$

Thus $\{U_n,\mathcal{A}_n\}_{n=1}^\infty$ is a reverse-martingale.
(b) By the reverse martingale convergence theorem it follows that $\{U_n\}$ is uniformly integrable and

$$U_n \rightarrow_{a.s.,1} E[h(X_1,X_2)|\mathcal{A}_\infty]\,.$$

But $\lim_n U_n$ is a tail random variable, and hence is a.s. constant by the Kolmogorov zero-one law. Thus $E[h(X_1,X_2)|\mathcal{A}_\infty]$ is a.s. constant, and since $E\{E[h(X_1,X_2)|\mathcal{A}_\infty]\} = Eh(X_1,X_2)$, it follows that $E[h(X_1,X_2)|\mathcal{A}_\infty] = Eh(X_1,X_2)$ a.s. Thus we conclude that

$$U_n \rightarrow_{a.s.,1} Eh(X_1,X_2)\,.$$

Remarks: While the condition $E|X_1|<\infty$ is necessary for the SLLN, the condition $E|h(X_1,X_2)|<\infty$ is *not necessary* for $U_n \rightarrow_{a.s.} Eh(X_1,X_2)$. See Giné and Zinn (1994), Zhang (1997), Latala and Zinn (1998), and de la Peña and Giné (1999), pages 160 - 164.

(c) For $h : R^k \to R$ (symmetric), the argument is essentially the same: for $k \le n$,

$$\begin{aligned}
E[h(X_1, \ldots, X_k)|\mathcal{A}_n] &= E[h(X_1, \ldots, X_n)|\underline{X}_{n:n}, X_{n+1}, X_{n+2}, \ldots] \\
&= E[h(X_1, \ldots, X_k)|\underline{X}_{n:n}] \\
&= \binom{n}{k}^{-1} \sum_{1 \le i_1 < \ldots < i_k \le n} E[h(X_{i_1}, \ldots, X_{i_k})|\underline{X}_{n:n}] \\
&= E[U_n|\underline{X}_{n:n}] \\
&\quad \text{by linearity of conditional expectation} \\
&\quad \text{and by definition of } U_n \\
&= U_n
\end{aligned}$$

since U_n is measurable with respect to $\sigma[\underline{X}_{n:n}]$. (the formula corresponding to (1) is now more complicated, though).

3. Exercise 18.5.1, PfS page 411. Suppose that $\{X_n, \mathcal{A}_n\}$ is a submartingale with Doob-decomposition given by $X_n = Y_n + A_n$ where $\{Y_n, \mathcal{A}_n\}$ is a martingale and A_n is a predictable process. Show that if $\{X_n\}$ is integrable, then $\{A_n\}$ is uniformly integrable.
Without loss of generality, suppose that $A_0 = EX_0 = 0$; if not, subtract EX_0 from X_n and A_n. Now $E(X_n) = E(A_n) \ge 0$, so

$$E(A_n) = |E(X_n)| \le E|X_n| \le \sup_n E|X_n| < \infty\,,$$

and hence $\sup_n EA_n \le \sup_n E|X_n| < \infty$; i.e. $\{A_n\}$ is integrable. To see that $\{A_n\}$ is uniformly integrable, note that $A_n \nearrow$ a.s., so $\lim_n A_n = A_\infty$ exists a.s., and by the monotone convergence theorem

$$E(A_\infty) \equiv E(\lim_n A_n) = \lim_n E(A_n) \le \sup E(A_n) < \infty\,.$$

By Vitali's theorem, this implies that $\{A_n\}$ is uniformly integrable.

4. Exercise 18.7.1, PfS page 422. Show that $E(Z_\tau^{(2)} = 0$, and $E(Z_\tau) = 1$ where $Z_n^{(2)} = S_n - n(p-q)$, $Z_n = (q/p)^{S_n}$, and $\tau = \inf\{n : S_n = -a \text{ or } b\}$ for a, b positive integers. First consider the bounded stopping

time $\tau \wedge k$. Then, by the basic optional sampling theorem it follows that

(2) $$0 = E(Z^{(2)}_{\tau\wedge k}) = E(S_{\tau\wedge k}) - E(\tau\wedge k)(p-q)\,.$$

Now $\tau \wedge k \nearrow \tau$ as $k \to \infty$, so $E(\tau\wedge k) \nearrow E(\tau)$ by the monotone convergence theorem. On the other hand, $S_{\tau(\omega)\wedge k}(\omega) \to S_{\tau(\omega)}(\omega)$ and $|S_{\tau\wedge k}| \le \max\{a,b\}$, so that by the dominated convergence theorem $E(S_{\tau\wedge k}) \to E(S_\tau)$. Therefore, taking limits across (2) yields

(3) $$0 = E(S_\tau) - E(\tau)(p-q) = E(Z^{(2)}_\tau)\,;$$

i.e. the first of the two identities holds. Similarly, by the basic optional sampling theorem it follows that

(4) $$E(Z_0) = 1 = E(Z_{\tau\wedge k})\,;$$

here we have $Z_{\tau\wedge k} \to Z_\tau$ as $k \to \infty$, while

$$|Z_{\tau\wedge k}| \le \max\{(q/p)^b, (p/q)^a\} < \infty\,,$$

so that the dominated convergence theorem yields $\lim_k E(Z_{\tau\wedge k}) = E(\lim_k Z_{\tau\wedge k}) = E(Z_\tau)$. Thus taking limits on k across (4) yields

$$E(Z_0) = 1 = E(Z_\tau)\,;$$

i.e. the second identity holds.

5. Exercise 18.7.2, PfS page 422. Suppose tht S_μ is Brownian motion with drift: $S_\mu(t) = S(t) + \mu t$ for $t \ge 0$. Let $\tau_{ab} \equiv \tau \equiv \inf\{t \ge 0 :\ S_\mu(t) = -a \text{ or } b\}$ where $-a < 0 < b$.
Claim 1: $S_0(t)$, $S_0^2(t) - t$, $S_\mu(t) - \mu t$ are mean 0 martingales, and, with $\theta = -2\mu$,

$$\exp(\theta[S_\mu(t) - \mu t] - \theta^2 t/2) = \exp(-2\mu[S(t) + \mu t])$$

is a mean 1 martingale.
Proof of claim 1: Since standard Brownian motion S has independent increments, with $\mathcal{A}_t \equiv \sigma[S(s), 0 \le s \le t]$ we have, for $0 \le s \le t$,

$$\begin{aligned} E(S(t)|\mathcal{A}_s) &= E(S(t) - S(s) + S(s)|\mathcal{A}_s) \\ &= E(S(t) - S(s)|\mathcal{A}_s) + E(S(s)|\mathcal{A}_s) \\ &= E(S(t) - S(s)) + S(s) = 0 + S(s) = S(s) \qquad \text{a.s.} \end{aligned}$$

so that $\{S(t), \mathcal{A}_t\}_{t\geq 0}^{\infty}$ is a zero - mean martingale. Since $S_\mu(t) - \mu t = S_0(t) = S(t)$, it follows immediately that $\{S_\mu(t) - \mu t, \mathcal{A}_t\}_{t\geq 0}^{\infty}$ is also a 0-mean martingale. To see that $\{S^2(t) - t, \mathcal{A}_t\}_{t\geq 0}^{\infty}$ is a zero-mean martingale, we calculate

$$\begin{aligned}
E(S^2(t) - t|\mathcal{A}_s) &= E([S(t) - S(s) + S(s)]^2 - (t - s + s)|\mathcal{A}_s) \\
&= E((S(t) - S(s))^2 - (t - s)|\mathcal{A}_s) \\
&\quad + E(2(S(t) - S(s))S(s)|\mathcal{A}_s) \\
&\quad + E(S^2(s) - s|\mathcal{A}_s) \\
&= E(S(t) - S(s)^2) - (t - s) + 2S(s)E(S(t) - S(s)) \\
&\quad + (S^2(s) - s) \\
&= 0 + 0 + S^2(s) - s = S^2(s) - s \qquad \text{a.s.}
\end{aligned}$$

so that the claim holds. (Note that this shows that $\langle S\rangle(t) = t$ is the predictable variation process corresponding to the sub - martingale $S^2(t)$.) To see that $Y_t = \exp(\theta[S_\mu(t) - \mu t] - \theta^2 t/2) = \exp(-2\mu[S(t) + \mu t])$ is a mean 1 martingale, note that $Y_t = \exp(\theta S(t) - \theta^2 t/2)$ and hence

$$\begin{aligned}
E(Y_t|\mathcal{A}_s) &= E(\exp(\theta(S(t) - S(s)))|\mathcal{A}_s) \cdot E(\exp(\theta S(s) - \theta^2 s/2)|\mathcal{A}_s) \\
&\qquad \cdot \exp(\theta^2(s/2 - t/2)) \\
&= E(\exp(\theta(S(t) - S(s)))) \cdot \exp(\theta^2(s/2 - t/2)) \cdot Y_s \qquad \text{a.s.} \\
&= \exp(\theta^2(t/2 - s/2)) \cdot \exp(\theta^2(s/2 - t/2)) \cdot Y_s \qquad \text{a.s.} \\
&= Y_s\,,
\end{aligned}$$

so that Y_t is a mean 1 mg. The second part of this holds simply because, with $\theta = -2\mu$ we have

$$\theta[S_\mu - \mu t] - \theta^2 t/2 = -2\mu S_\mu + 2\mu^2 t - 4\mu^2 t/2 = -2\mu S_\mu(t)\,.$$

Claim 2: If $\mu = 0$, $P(S(\tau) = -a) = b/(a + b)$ and $E\tau = ab$.
Claim 3: If $\mu \neq 0$, then

$$P(S(\tau) = -a) = \frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}}$$

and

$$E(\tau) = \frac{b}{\mu} - \frac{a + b}{\mu}\frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}}\,.$$

To prove claim 2, first consider the bounded stopping times $\tau \wedge k$. Then by the basic optional sampling theorem,

$$0 = E(S_0^2(\tau \wedge k) - \tau \wedge k)\,. \tag{5}$$

Now $\tau \wedge k \nearrow \tau$, so that $E(\tau \wedge k) \to E(\tau)$ by the monotone convergence theorem, while $S_0(\tau \wedge k) \to S_0(\tau)$ with $|S_0(\tau \wedge k)| \leq a \vee b < \infty$ for all k, and hence $E(S_0^2(\tau \wedge k)) \to E(S_0^2(\tau))$ by the dominated convergence theorem. Thus taking limits across (5) yields

$$E(S_0^2(\tau)) = E(\tau)\,,$$

and when $\mu = 0$, this implies that $E(\tau) < \infty$. By playing this game with the martingale S, we find that $E(S(\tau \wedge k)) = 0$, and by the dominated convergence theorem, $E(S(\tau)) = 0$. Since $S(\tau)$ takes on the two values $-a$ and b, we have

$$0 = E_0 S(\tau) = -aP_0(S(\tau) = -a) + bP_0(S(\tau) = b) = -a(1 - p_b) + bp_b$$

so that $p_b = a/(b+a)$, $p_a = 1 - p_b = b/(b+a)$. From $E(S_0^2(\tau)) = E(\tau)$ it then follows that

$$E(\tau) = a^2 p_a + b^2 p_b = a^2 \frac{b}{b+a} + b^2 \frac{a}{b+a} = ab\,,$$

completing the proof of Claim 2.
Proof of Claim 3. Similarly, when $\mu \neq 0$, the basic optional sampling theorem yields

$$0 = E(S_\mu(\tau \wedge k) - (\tau \wedge k)\mu)\,. \tag{6}$$

Now $\tau \wedge k \nearrow \tau$, so that $E(\tau \wedge k) \to E(\tau)$ by the monotone convergence theorem, while $S_\mu(\tau \wedge k) \to S_\mu(\tau)$ with $|S_\mu(\tau \wedge k)| \leq a \vee b < \infty$ for all k, and hence $E(S_\mu(\tau \wedge k)) \to E(S_\mu(\tau))$ by the dominated convergence theorem. Thus taking limits across (6) yields

$$E(S_\mu(\tau)) = \mu E(\tau)\,,$$

and this implies that $E(\tau) < \infty$ for $\mu \neq 0$. Again the basic optional sampling theorem implies that

$$E(Y(0)) = 1 = E \exp(-2\mu S_\mu(\tau \wedge k))\,,$$

for each k, and by the dominated convergence theorem this yields

$$\begin{aligned}
E(Y(0)) = 1 &= E\exp(-2\mu S_\mu(\tau)) \\
&= P(S_\mu(\tau) = -a)\exp(2\mu a) + P(S_\mu(\tau) = b)\exp(-2\mu b) \\
&= p_a \exp(2\mu a) + (1 - p_a)\exp(-2\mu b) \\
&= p_a(\exp(2\mu a) - \exp(-2\mu b)) + \exp(-2\mu b)
\end{aligned}$$

so that

$$p_a = \frac{1 - \exp(-2\mu b)}{\exp(2\mu a) - \exp(-2\mu b)} = \frac{1 - \exp(2\mu b)}{1 - \exp(2\mu(a+b))}.$$

Then, finally, since $E(S_\mu(\tau)) = \mu$,

$$\begin{aligned}
E(\tau) &= \frac{E(S_\mu(\tau))}{\mu} \\
&= \frac{1}{\mu}\left\{-ap_a + b(1 - p_a)\right\} \\
&= \frac{1}{\mu}\left\{b - (a+b)\frac{1 - \exp(2\mu b)}{1 - \exp(2\mu(a+b))}\right\}.
\end{aligned}$$

Note that when $\mu < 0$ we have

$$\begin{aligned}
P(\|S_\mu\|_0^\infty \geq b) &= \lim_{a\to\infty} P(\tau_{ab} < \infty) \\
&= \lim_{a\to\infty} P(S_\mu(\tau_{ab}) = b) \\
&= \exp(-2|\mu|b)
\end{aligned}$$

so that $\|S_\mu\|_0^\infty \sim \text{Exponential}(2|\mu|)$.

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