Statistics 522, Problem Set 6 Solutions

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1. Exercise 18.3.5, PfS page 403. Let $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ be a sub-martingale with $X_n \ge 0$. Let $r > 1$. Then $\{X_n^r\}$ is uniformly integrable if and only if $\{X_n^r\}$ is integrable.

   Solution: Uniform integrability implies integrability, so it remains only to prove the reverse implication. Suppose that $\{X_n^r\}$ is integrable. Then $\{X_n\}$ is uniformly integrable, and hence by the s-martingale convergence theorem 18.3.1, $X_n \to X_\infty \in L_1$ where $\{X_n, \mathcal{A}_n\}_{n=0}^{\overline{\infty}}$ is a sub-mg; i.e. $E(X_\infty|\mathcal{A}_n) \ge X_n$ a.s. and

   $$E(X_\infty^r) = E(\liminf X_n^r) \le \liminf E(X_n^r) \le \sup_n E(X_n^r) < \infty$$

   by Fatou's lemma and integrability of $\{X_n^r\}$. Hence by the conditional Jensen inequality,

   $$E(X_n^r) \le E\{E(X_\infty|\mathcal{A}_n)^r\} \le E\{E(X_\infty^r|\mathcal{A}_n)\} = E(X_\infty^r)$$

   and it follows from Vitali's theorem that $\{X_n^r\}$ is uniformly integrable.

   Alternatively, by Doob's L_r-maximal inequality, since $\{X_n, \mathcal{A}_n\}$ is a sub-martingale,

   $$E\left\{\left(\max_{1\le k\le n} X_k\right)^r\right\} \le \left(\frac{r}{r-1}\right)^r E|X_n|^r\,,$$

   and hence, by the monotone convergence theorem,

   $$E[\sup_{1\le k<\infty} X_k^r] \le \left(\frac{r}{r-1}\right)^r \sup_n E|X_n|^r < \infty\,.$$

   Thus with $Y \equiv \sup_{1\le k<\infty} X_k$, it follows that

   $$\sup_n E\left\{X_n^r 1_{[X_n^r\ge\lambda]}\right\} \le E(Y^r 1_{[Y^r\ge\lambda]} \to 0$$

   as $\lambda \to \infty$; i.e. $\{X_n^r\}$ is uniformly integrable.

2. Exercise 18.3.7, PfS page 403. Let $r > 1$. Let $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ be a martingale. Then the following are equivalent:
(10) The $|X_n|^r-$process is integrable.
(11) $X_n \to_r X_\infty$
(12) The X_n's are uniformly integrable (thus $X_n \to$ (some X_∞) a.s.) and $X_\infty \in L_r$.
(13) The $|X_n|^r$'s are uniformly integrable.
(14) $\{|X_n|^r, \mathcal{A}_n\}_{n=0}^{\overline{\infty}}$ is a submg and $E|X_n|^r \nearrow E|X_\infty|^r$.

Solution: Suppose that (10) holds. Then $|X_n|^r$ is an integrable submg. Thus the $|X_n|^r$ are uniformly integrable by the preceding problem. Thus (13) holds.
Suppose (13) holds. Then $\{X_n\}$ is uniformly integrable, and $X_n \to_{a.s.} X_\infty \in L_1$ and

$$E|X_\infty|^r = E(\liminf |X_n|^r) \leq \liminf E|X_n|^r \leq \sup_n E|X_n|^r < \infty\,,$$

so $X_\infty \in L_r$; i.e. (12) holds.
Suppose (12) holds. Then $\{|X_n|, \mathcal{A}_n\}_{n=0}^{\overline{\infty}}$ is a sub-martingale by Theorem 16.3.1. Thus $|X_n| \leq E(|X_\infty| \mathcal{A}_n)$, so $|X_n|^r \leq \{E(|X_\infty||\mathcal{A}_n)\}^r \leq E(|X_n|^r|\mathcal{A}_n)$ a.s., and hence $E|X_n|^r \leq E|X_\infty|^r < \infty$; i.e. (10) holds.
Thus (10) iff (12) iff (13) holds.
Now (11) implies (10) since

$$E|X_n|^r \leq c_r\{E|X_n - X_\infty|^r + E|X_\infty|^r\}$$

by the c_r-inequality.
Suppose that (13) holds. Then $X_n \to_{a.s.} X_\infty \in L_r$ (by (13) implies (12)), and since $\{|X_n|^r, \mathcal{A}_n\}_{n=0}^{\overline{\infty}}$ is a sub-mg,

$$\limsup_{n\to\infty} E|X_n|^r \leq E|X_\infty|^r < \infty\,.$$

Hence $X_n \to_r X_\infty$ by Vitali's theorem; i.e. (11) holds. Thus (10) iff (12) iff (13) iff (14).