Statistics 522, Problem Set 5 Solutions

Wellner; 2/12/99

1. Exercise 18.1.5, PfS page 398: Find the exponential martingale corresponding to $M(t)$ in example 1.12. Then differentiate this twice with respect to c, setting $c = 0$ each time, to obtain the mg's given in the example.

   Solution: In example 1.12, $N(t)$ is a Poisson process with intensity λ; and $M(t) = N(t) - \lambda t$ and $M^2(t) - \lambda t$ are both martingales. A natural exponential martingale to consider is

   $$Y(t) \equiv Y_c(t) \equiv \frac{\exp(cM(t))}{E\exp(cM(t))}\,.$$

   Since

   $$\begin{aligned} E\exp(cM(t)) &= E\exp(c(N(t) - \lambda t)) \\ &= \exp(-c\lambda t)E\exp(c(N(t)) \\ &= \exp(-c\lambda t)\exp((e^c - 1)\lambda t)\,, \end{aligned}$$

   we find that

   $$Y(t) = \exp(cN(t) - (e^c - 1)\lambda t)\,.$$

   I claim that $\{Y(t), \mathcal{A}_t\}_{t=0}^{\infty}$ is a martingale on $[0, \infty)$. To see this, note that for $0 \le s < t < \infty$ we have

   $$\begin{aligned} E\left(Y(t)|\mathcal{A}_s\right) &= E\left(\exp(cN(t) - (e^c - 1)\lambda t)|\mathcal{A}_s\right) \\ &= E\left(\exp(c(N(t) - N(s)) - (e^c - 1)\lambda(t - s))\exp(cN(s) - (e^c - 1)\lambda s)|\mathcal{A}_s\right) \\ &= Y(s)E\left(\exp(c(N(t) - N(s)) - (e^c - 1)\lambda(t - s))|\mathcal{A}_s\right) \qquad \text{a.s.} \\ &= Y(s)E\left(\exp(c(N(t) - N(s)) - (e^c - 1)\lambda(t - s))\right) \qquad \text{a.s.} \\ &\qquad\qquad \text{since } \ N(t) - N(s) \ \text{ is independent of } \ \mathcal{A}_s \\ &= Y(s)\cdot 1 = Y(s) \qquad \text{a.s.}\,, \end{aligned}$$

   and hence $\{Y(t), \mathcal{A}_t\}_{t=0}^{\infty}$ is a martingale. Note that

   $$\begin{aligned} Y_c'(t) \equiv \frac{d}{dc}Y_c(t)|_{c=0} &= Y_c(t)\left(N(t) - e^c\lambda t\right)|_{c=0} \\ &= Y_0(t)\left(N(t) - e^c\lambda t\right)|_{c=0} \\ &= N(t) - \lambda t = M(t)\,, \end{aligned}$$

while

$$\begin{aligned} Y_c''(t) \equiv \frac{d^2}{dc^2} Y_c(t)|_{c=0} &= Y_c(t)\,(N(t) - e^c \lambda t)^2\,|_{c=0} + Y_c(t)\,(-e^c \lambda t)\,|_{c=0} \\ &= Y_0(t)\,(N(t) - e^c \lambda t)\,|_{c=0} \\ &= M^2(t) - \lambda t\,. \end{aligned}$$

2. Exercise 18.3.3, PfS page 403. (a) Let $\cdots \subset \mathcal{A}_{-1} \subset \mathcal{A}_0 \subset \cdots$ be sub $\sigma-$fields of $\mathcal{A}$. Let $X \in \mathcal{L}_1(\Omega, \mathcal{A}, P)$. Let $Y_n \equiv E(X|\mathcal{A}_n)$. Then $(Y_n, \mathcal{A}_n)_{n=-\infty}^{\infty}$ is a uniformly integrable mg.

   Solution: Now

   $$P(|Y_n| \geq \lambda) \leq \frac{E|Y_n|}{\lambda} = \frac{E|E(X|\mathcal{A}_n)|}{\lambda} \leq \frac{E|X|}{\lambda}$$

   so that

   $$\sup_{-\infty<n<\infty} P(|Y_n| \geq \lambda) \leq \frac{E|X|}{\lambda} \to 0$$

   as $\lambda \to \infty$. Thus we have

   $$E|Y_n| 1_{[|Y_n| \geq \lambda]} = E|E(X|\mathcal{A}_n)| 1_{[|Y_n| \geq \lambda]} \leq E|X| 1_{[|Y_n| \geq \lambda]} \to 0$$

   uniformly in n by the absolute continuity of the integral. Thus $(Y_n, \mathcal{A}_n)_{n=-\infty}^{\infty}$ is a uniformly integrable mg.

3. Exercise E10.3, PwMG, page 232. Suppose that S and T are stopping times relative to $(\Omega, \mathcal{F}, \mathcal{F}_n)$. Prove that $S \wedge T$, $S \vee T$, and $S + T$ are stopping times.

   Solution: To see that $S \wedge T$ is a stopping time, note that

   $$[S \wedge T > n] = [S > n] \cap [T > n] = [S \leq n]^c \cap [T \leq n]^c$$

   where $[S \leq n] \in \mathcal{F}_n$ and $[T \leq n] \in \mathcal{F}_n$ since S and T are stopping times. It follows that $[S \wedge T > n] \in \mathcal{F}_n$, and hence also $[S \wedge T \leq n] \in \mathcal{F}_n$; i.e. $S \wedge T$ is a stopping time.
   To see that $S \vee T$ is a stopping time, note that

   $$[S \vee T \leq n] = [S \leq n] \cap [T \leq n]$$

where $[S \leq n] \in \mathcal{F}_n$ and $[T \leq n] \in \mathcal{F}_n$. It follows that $[S \vee T \leq n] \in \mathcal{F}_n$; i.e. $S \vee T$ is a stopping time.
To see that $S+T$ is a stopping time, note that

$$[S+T \leq n] = \cup_{k=0}^{n} [S=k] \cap [T = n-k]$$

where $[S = k] \in \mathcal{F}_k \subset \mathcal{F}_n$ and where $[T = n-k] \in \mathcal{F}_{n-k} \subset \mathcal{F}_n$, for $k = 0, \ldots, n$. Hence $[S+T \leq n] \subset \mathcal{F}_n$; i.e. $S+T$ is a stopping time.

4. Exercise E10.1, PwMG, page 231. Polyá's urn: At time 0, an urn contains 1 black ball and 1 white ball. At each time $1, 2, 3, \ldots$, a ball is chosen at random from the urn, and is replaced together with a new ball of the same color. Just after time n, there are therefore $n+2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. Let $M_n = (B_n+1)/(n+2)$, the proportion of black balls in the urn just after time n. Prove that (relative to a natural filtration which you should specify) M is a martingale. Prove that $P(B_n = k) = 1/(n+1)$ for $0 \leq k \leq n$. What is the distribution of $\Theta \equiv \lim_n M_n$? Prove that for $0 < \theta < 1$,

$$N_n^\theta \equiv \frac{(n+1)!}{B_n!(n-B_n)!} \theta^{B_n} (1-\theta)^{n-B_n}$$

defines a martingale N^θ.

Solution: Let $\mathcal{F}_n \equiv \sigma(B_1, \ldots, B_n)$. Note that $M_n \equiv (B_n+1)/(n+2)$ is the conditional (given $\mathcal{F}_n$) probability of drawing a black ball at the $n+1$st draw. Thus we compute

$$\begin{aligned} E(M_{n+1}|\mathcal{F}_n) &= E\left(\frac{B_{n+1}+1}{n+3}|\mathcal{F}_n\right) = \frac{1}{n+3} E(B_{n+1}+1|\mathcal{F}_n) \\ &= \frac{1}{n+3}\{(B_n+1)(1-M_n) + (B_n+2)M_n\} \\ &= \frac{1}{n+3}\{B_n + 1 - M_n + 2M_n\} \\ &= \frac{1}{n+3}\{(n+2)M_n + M_n\} = M_n \qquad \text{a.s.}. \end{aligned}$$

Hence $\{M_n, \mathcal{F}_n\}$ is a martingale. Similarly, letting

$$p_n(k) \equiv \frac{(n+1)!}{k!(n-k)!}\theta^k(1-\theta)^{n-k},$$

the process $N_n^\theta = p_n(B_n)$ and

$$\begin{aligned} E(N_{n+1}^\theta|\mathcal{F}_n) &= E(p_{n+1}(B_{n+1})|\mathcal{F}_n) \\ &= p_{n+1}(B_n)(1-M_n) + p_{n+1}(B_n+1)M_n \\ &= \frac{(n+2)!}{B_n!(n+1-B_n)!}\theta^{B_n}(1-\theta)^{n+1-B_n}\frac{(n+1-B_n)}{(n+2)} \\ &\qquad + \frac{(n+2)!}{(B_n+1)!(n+1-B_n-1)!}\theta^{B_n+1}(1-\theta)^{n+1-B_n-1}\frac{(B_n+1)}{(n+2)} \\ &= \frac{(n+1)!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}\{(1-\theta)+\theta\} \\ &= p_n(B_n) \equiv N_n^\theta \qquad \text{a.s.,} \end{aligned}$$

so $\{N_n^\theta, \mathcal{F}_n\}$ is a martingale. This implies that $EN_n^\theta = EN_0^\theta = 1$ for all $\theta \in (0,1)$, or

$$E\left\{\frac{n!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}\right\} = \frac{1}{n+1}\,. \tag{1}$$

This equality clearly holds if $P(B_n = k) = 1/(n+1)$ for $k = 0, \ldots, n$. On the other hand, (1.1) implies, by letting $\alpha = \theta/(1-\theta)$, that, with $p_k = P(B_n = k)$,

$$\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\alpha^k p_k = \frac{1}{n+1}(1+\alpha)^n = \frac{1}{n+1}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\alpha^k\,,$$

and this yields $p_k = 1/(n+1)$ by matching coefficients.

The distribution of B_n is a discrete uniform distribution on $0, \ldots, n$ for every n, so the distribution of M_n is a discrete uniform distribution on $0 < 1/(n+1) < \cdots < (n+1)/(n+2) < 1$ and it is clear that $M_n \to_d U(0,1)$ as $n \to \infty$; $P(M_n \le u) = [(n+2)u]/(n+1) \to u = P(U \le u)$ where $U \sim$Uniform$(0,1)$.

5. Exercise E10.7, PwMG, page 233. Let $X_1, X_2, \ldots$ be i.i.d rv's with $P(X = 1) = p$, $P(X = -1) = 1 - p \equiv q$, where $0 < p < 1$ and $p \neq q$. Suppose that a, b are integers with $0 < a < b$. Define

$$S_n = a + X_1 + \ldots + X_n, \qquad T \equiv \inf\{n : S_n = 0, \text{or} \quad S_n = b\}.$$

Let $\mathcal{F}_n \equiv \sigma[X_1, \ldots, X_n]$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Explain why T satisfies the condition in Question E10.5. Prove that $M_n \equiv (q/p)^{S_n}$ and $N_n = S_n - n(p - q)$ define martingales M and N. Deduce the values of $P(S_T = 0)$ and $E(S_T)$.

Solution: T is clearly a stopping time and, for each n

$$P(T \leq n + b | \mathcal{F}_n) \geq p^{b - S_n} + q^{S_n} \geq (p \wedge q)^b \equiv \epsilon > 0$$

since $p \in (0, 1)$. Thus the hypotheses of E10.5 hold with $N = b$ and $\epsilon \equiv (p \wedge q)^b$. Thus $E(T) < \infty$, and the third set of sufficient conditons for Doob's optional sampling theorem hold. Since $\{S_n - n(p - q), \mathcal{F}_n\}$ and $\{(q/p)^{S_n}, \mathcal{F}_n\}$ are both martingales, we conclude from Doob's optional sampling theorem that

$$(2) \qquad E\left(\frac{q}{p}\right)^{S_T} = E\left(\frac{q}{p}\right)^{S_0} = E\left(\frac{q}{p}\right)^{a}.$$

But the left side of (2) equals

$$\left(\frac{q}{p}\right)^{b} P(S_T = b) + \left(\frac{q}{p}\right)^{0} P(S_T = 0) \equiv \left(\frac{q}{p}\right)^{b} p_b + \left(\frac{q}{p}\right)^{0} (1 - p_b).$$

Thus we can solve for p_b to obtain

$$p_b = P(S_T = b) = \frac{(q/p)^a - 1}{(q/p)^b - 1},$$

and

$$p_0 = P(S_T = 0) = 1 - p_b = \frac{(q/p)^b - (q/b)^a}{(q/p)^b - 1}.$$

It follows easily that

$$E(S_T) = b p_b + 0 p_0 = b\frac{(q/p)^a - 1}{(q/p)^b - 1}.$$

You should also take a look at the situation for $p = q = 1/2$ in Section 18.7, pages 421 - 422.