Statistics 522, Problem Set 3 Solutions

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1. Let $X_1, X_2, \ldots$ be iid where $P(X_k = 0) = P(X_k = 2) = 1/2$ for all k. Show that $\sum_{n=1}^{\infty} X_n/3^n \rightarrow_{a.s.}$ (some S), and determine the mean, variance, and the name of the df F_S of S.

   Solution: Let $Y_n = X_n/3^n$ where $P(X_n = 0) = 1/2 = P(X_2 = 2)$. Then we compute $\mu_n = EY_n = 0 \cdot (1/2) + (2/3^n)(1/2) = (1/3^n)$, $\sigma_n^2 = Var(Y_n) = 1/3^{2n}$, so that

$$\sum_{k=1}^{n} \mu_k = \sum_{k=1}^{n} \frac{1}{3^k} \rightarrow \sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1}{2}$$

   and

$$\sum_{k=1}^{n} \sigma_k^2 = \sum_{k=1}^{n} \frac{1}{3^{2k}} \rightarrow \sum_{k=1}^{\infty} \frac{1}{3^{2k}} = \frac{1}{8}.$$

   Hence, by the 2-series theorem,

$$S_n = \sum_{k=1}^{n} Y_k \rightarrow_{a.s.} \sum_{k=1}^{\infty} Y_k = S$$

   and we have $E(S) = 1/2$, $Var(S) = 1/8$. By the construction it is clear that S_n takes on values in the Cantor set C for each n and so does S: $P(S \in C) = 1$. Setting $F_n(x) \equiv P(S_n \le x)$ for $x \in (0,1)$, we find that

$$F_1(x) = \begin{cases} 1/2 & 0 \le x < 2/3 \\ 1 & 2/3 \le x < 1 \end{cases},$$

$$F_2(x) = \begin{cases} 1/4 & 0 \le x < 2/9 \\ 1/2 & 2/9 \le x < 2/3 \\ 3/4 & 2/3 \le x < 7/9 \end{cases},$$

   and, in general, $F_n(x) = \sum_{k=1}^{n} a_k/2^{k+1}$ for $x = \sum_{k=1}^{n} a_k/3^k$, $a_k \in \{0, 2\}$. Since $S_n \rightarrow_{a.s.} S$ implies that $S_n \rightarrow_d S$, we conclude that for

$x = \sum_{k=1}^{\infty} a_k/3^k$, $a_k \in \{0, 2\}$,

$$\begin{aligned} F_S(x) &= \lim_n F_n(x) = \lim_{n\to} P(S_n \le x) = P(S \le x) \\ &= \sum_{k=1}^{\infty} a_k/2^{k+1}\,, \end{aligned}$$

and hence the distribution F_S of S is the Cantor singular distribution on $[0, 1]$.

2. Suppose that $X_k \sim$Uniform$[-k, k]$ for $k \ge 1$, and where $0 < a < 1$. (a) Show that $\sum_{k=1}^{n} a^k X_k \to_{a.s.}$(some S).
(b) Evaluate the mean and variance of S.

Solution: (a) Let $Y_k \equiv a^k X_k$. Then the Y_k's are independent with $\mu_k = 0$, $Var(Y_k) = a^{2k}k^2/3$. Thus we have $\mu = \sum_{n=1}^{\infty} \mu_n = 0$, and

$$\sigma^2 = \sum_{n=1}^{\infty} \sigma_n^2 = \frac{1}{3}\sum_{n=1}^{\infty} n^2 a^{2n} < \infty$$

Now

$$\begin{aligned} |Y_k| &\le \sup_{k\ge1} ka^k \le \sup_{x\ge1} xa^x \\ &\le \sup_{x>0} x\exp(-x\log(1/a)) = \frac{e^{-1}}{\log(1/a)}\,. \end{aligned}$$

Thus the bounded part of the two-series theorem applies and we conclude that $S_n \equiv \sum_{k=1}^{n} Y_k \to_{a.s.} S$ with $E(S) = 0$ and $Var(S) = (1/3)\sum_{n=1}^{\infty} n^2 a^{2n} < \infty$.
(b) From part (a), $E(S) = 0$ and

$$Var(S) = \frac{1}{3}\sum_{n=1}^{\infty} n^2 a^{2n} = \frac{1}{3}\frac{a^2(a^2+1)}{(1-a^2)^3}\,,$$

by differentiating twice across the identity

$$\sum_{k=0}^{\infty} a^{2k} = \frac{1}{1-a^2}\,.$$

3. If $X_1, X_2, \ldots$ are arbitrary rv's with all $X_k \geq 0$ a.s., then $\sum_{k=1}^{\infty} EX_k < \infty$ implies that $\sum_{k=1}^{\infty} X_k \to_{a.s.}$ (some S).

   Solution: Let $S_n = \sum_{i=1}^{n} X_i$. Then since all the X_k's are ≥ 0, for $k > n$ we have $S_k - S_n = \sum_{j=n+1}^{k} X_j \leq \sum_{j=n+1}^{m} X_j$ a.s. so that $\max_{n<k\leq m}(S_k - S_n) = S_m - S_n$. Therefore it follows that for every $\epsilon > 0$ we have

$$\begin{aligned} P(\max_{n<k\leq m} |S_k - S_n| \geq \epsilon) &= P(S_m - S_n \geq \epsilon) \leq \frac{E(\sum_{j=n+1}^{m} X_j)}{\epsilon} \\ &\leq \frac{\sum_{j=n+1}^{\infty} E(X_j)}{\epsilon} \\ &\to 0 \end{aligned}$$

   as $m, n \to \infty$ since $\sum_{j=1}^{\infty} E(X_j) < \infty$.

4. Suppose that $P(A_n \text{ i.o.}) = 1$ and $P(B_n^c \text{ i.o.}) = 0$. Show that $P(A_n \cap B_n \text{ i.o.}) = 1$.

   Solution: Note that for $m \geq n$, $B_m \supset \cap_{k=n}^{\infty} B_k$, and hence

$$\begin{aligned} [A_n \cap B_n \text{ i.o.}] &= \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m \cap B_m \\ &= \lim_{n\to\infty} \cup_{m=n}^{\infty} A_m \cap B_m \\ &\supset \lim_{n\to\infty} \cup_{m=n}^{\infty} A_m \cap \cap_{k=n}^{\infty} B_k \\ &= \lim_{n\to\infty} \{\cup_{m=n}^{\infty} A_m \cap \cap_{k=n}^{\infty} B_k\} \\ &= [A_n \text{ i.o.}] \cap [B_n \text{ a.a.}] \end{aligned}$$

   where $P(A_n \text{ i.o.}) = 1$ and $P(B_n \text{ a.a.}) = P([B_n \text{ i.o.}]^c) = 1$. It follows that $P([A_n \cap B_n \text{ i.o.}]) = 1$.

5. Let X, X_1, $X_2, \ldots$ be i.i.d. and consider the partial sums $S_n = X_1 + \ldots + X_n$. Let $0 < r < 2$ (and suppose $E(X) = 0$ in case $1 \leq r < 2$). Show that the following are equivalent:
   (a) $E|X|^r < \infty$.
   (b) $S_n / n^{1/r} \to_{a.s.} 0$.
   (c) $E|S_n|^r = o(n)$.
   (d) $E(\max_{1\leq k\leq n} |S_k|^r) = o(n)$.
   (e) $n^{-1} \max_{1\leq k\leq n} |X_k|^r \to_{a.s.} 0$.

Solution: The equivalence of (a) and (b) is the Marcinkiewicz-Zygmund theorem (Exercise 10.4.4, page 188). Furthermore, from our proof of the SLLN, we know that (a) is equivalent to (e).

Now

$$n^{-1} \max_{k \le n} |X_k|^r \le n^{-1} \sum_{k=1}^{n} |X_k|^r \rightarrow_{a.s.} E|X|^r$$

where $E(n^{-1} \sum_{k=1}^{n} |X_k|^r) = E|X|^r$, so by Vitali's theorem $\{n^{-1} \max_{k \le n} |X_k|^r\}$ is uniformly integrable, and hence by (a) and (e) we have

$$E\left\{ n^{-1} \max_{k \le n} |X_k|^r \right\} \to 0 \,. \tag{1}$$

Suppose that (a) holds so that (b) and (e) also hold. Furthermore, (b) implies that $S_n/n^{1/r} \rightarrow_p 0$. By the Hoffmann-Jørgensen inequality, we know that

$$E \max_{k \le n} |S_k|^r \le K\{E \max_{1 \le i \le n} |X_i|^r + t_0^r\}$$

where

$$E \max_{1 \le i \le n} |X_i|^r = o(n)$$

by (1), and

$$t_0 = \inf\{t : P(\max_{k \le n} |S_k| > t) \le 1/(2 \cdot 4^r)\}.$$

Now

$$\begin{aligned} t_0^r/n &= \inf\{t/n^{1/r} : P(\max_{k \le n} |S_k| > t) \le 1/(2 \cdot 4^r)\}^r \\ &= \inf\{s > 0 : P(\max_{k \le n} |S_k| > s n^{1/r}) \le 1/(2 \cdot 4^r)\}^r \\ &\to 0 \end{aligned}$$

since, by the Ottaviani-Skorokhod inequality

$$P(\max_{1 \le k \le n} |S_k| > s n^{1/r}) \le \frac{P(|S_n| > s n^{1/r}/2)}{1 - \max_{1 \le k \le n} P(|S_n - S_k| > s n^{1/r}/2)}$$

where the numerator converges to 0 for every $s > 0$ and the denominator converges to 1 for every $s > 0$ since, for each $n_0 \leq n$,

$$\begin{aligned}
&\max_{1 \leq k \leq n} P(|S_n - S_k| > sn^{1/r}/2) \\
&\quad \leq \max_{n-n_0 < k \leq n} P(|S_n - S_k| > sn^{1/r}/2) + \max_{1 \leq k \leq n-n_0} P(|S_n - S_k| > sn^{1/r}/2) \\
&\quad \leq \max_{1 \leq j < n_0} P(|S_j| > sn^{1/r}/2) + \max_{n_0 \leq j \leq n} P(|S_j| > sn^{1/r}/2) \\
&\quad \leq \max_{1 \leq j < n_0} \frac{E|S_j|^r}{s^r n/2^r} + \max_{n_0 \leq j \leq n} P(|S_j| > sj^{1/r}/2) \\
&\quad \to 0 + 0
\end{aligned}$$

since the first term converges to zero for each fixed n_0 as $n \to \infty$ because $E|X|^r < \infty$ and the second term converges to 0 for each $s > 0$ as $n_0 \to \infty$ since $S_n/n^{1/r} \to_p 0$. Thus (a) implies (d).

(d) implies (c) trivially. On the other hand, (c) implies (a) easily as follows: suppose that (c) holds but (a) fails; i.e. $E|X|^r = \infty$. But since (c) holds, the C_r−inequality implies that

$$\begin{aligned}
\infty = E|X_n|^r &= E|S_n - S_{n-1}|^r \\
&\leq C_r \left\{ E|S_n|^r + E|S_{n-1}|^r \right\} \\
&= C_r o(n),
\end{aligned}$$

which is a contradiction. Hence $E|X_n|^r = E|X|^r < \infty$ and (c) implies that (a) holds.