Statistics 522, Problem Set 2 Solutions

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1. Suppose that $X_1, X_2, \ldots$ are independent random variables with $P(X_n = n^2 - 1) = 1/n^2$, $P(X_n = -1) = 1 - 1/n^2$. Show that $E(X_n) = 0$ for all n (so the game is "fair" for every n), but that

$$\overline{X}_n = n^{-1} S_n = n^{-1} \sum_{i=1}^{n} X_i \rightarrow_{a.s.} -1$$

as $n \rightarrow \infty$.

Solution: Note that $P(X_n = n^2 - 1) = P(X_n \neq -1) = n^{-2}$, and therefore it follows that $\sum_n (P(X_n \neq -1) < \infty$, and by Borel-Cantelli, $P(X_n \neq -1 \text{ i.o.}) = 0$. Hence $P(X_n = 1 \quad \text{a.a.}) = 1$ (here I'm using $[A_n \text{ a.a.}]$ for $\underline{\lim}_n A_n = (\overline{\lim}_n A_n^c)^c$) and hence $\overline{X}_n \rightarrow_{a.s.} -1$, even though

$$E(X_n) = (n^2 - 1) n^{-2} + (-1)(1 - n^{-2}) = 0$$

for all n.

2. Suppose that $X_1, X_2, \ldots$ are i.i.d. Exp(1). Show that
(a) $\overline{\lim}_n X_n / \log n = 1$ a.s., and
(b) $\lim_n X_{n:n} / \log n = 1$ a.s.

Solution: For any $\epsilon > 0$

$$P(X_n > (1+\epsilon)\log n) = e^{-(1+\epsilon)\log n} = 1/n^{1+\epsilon}$$

so that $\sum_{n=1}^{\infty} P(X_n > (1+\epsilon)\log n) < \infty$ and hence, by Borel-Cantelli, $P(X_n > (1+\epsilon)\log n \text{ i.o.}) = 0$. But this implies

$$\overline{\lim}_n X_n / \log n \leq 1 + \epsilon \qquad a.s.$$

for every $\epsilon > 0$, and hence

$$\overline{\lim}_n X_n / \log n \leq 1 \qquad a.s. \tag{1}$$

Furthermore,

$$P(X_n > (1-\epsilon)\log n) = e^{-(1-\epsilon)\log n} = 1/n^{1-\epsilon}$$

so that $\sum_{n=1}^{\infty} P(X_n > (1-\epsilon)\log n) = \infty$, and hence, by the second Borel-Cantelli lemma, $P(X_n > (1-\epsilon)\log n \text{ i.o.}) = 1$. This implies that

$$\overline{\lim}_n X_n/\log n \geq 1-\epsilon \qquad a.s.$$

for every $\epsilon > 0$, and hence

$$\overline{\lim}_n X_n/\log n \geq 1 \qquad a.s. \tag{2}$$

Combining (1) and (2) yields (a).

To prove (b), first note that for any fixed $n_0 \leq n$

$$\begin{aligned} \frac{X_{n:n}}{\log n} &= \max\left\{\frac{\max_{k\leq n_0} X_k}{\log n}, \frac{\max_{n_0<k\leq n} X_k}{\log n}\right\} \\ &\leq \max\left\{\frac{\max_{k\leq n_0} X_k}{\log n}, \max_{n_0<k\leq n}\frac{X_k}{\log k}\right\} \\ &\to_{a.s.} \max\left\{0, \sup_{n_0<k<\infty}\frac{X_k}{\log k}\right\} = \sup_{n_0<k<\infty}\frac{X_k}{\log k} \end{aligned}$$

where (recall (1.1.15), page 7),

$$\lim_{n_0\to\infty}\sup_{n_0<k<\infty}\frac{X_k}{\log k} = \limsup_{n\to\infty}\frac{X_n}{\log n} = 1$$

a.s. by part (a). This implies that

$$\limsup_{n\to\infty}\frac{X_{n:n}}{\log n} \leq 1 \qquad a.s. \tag{3}$$

On the other hand, using the fact that the X_i's are i.i.d.,

$$\begin{aligned} P(X_{n:n} \leq y) &= P(X_1 \leq y, \ldots, X_n \leq y) = P(X_1 \leq y)^n \\ &= (1-e^{-y})^n . \end{aligned}$$

Thus, since $1 - x \leq e^{-x}$,

$$P(X_{n:n} \leq (1-\epsilon)\log n) = \left(1 - \frac{1}{n^{1-\epsilon}}\right)^n \leq \{\exp(-n^{-1+\epsilon})\}^n = \exp(-n^{\epsilon})$$

so that $\sum_{n=1}^{\infty} P(X_{n:n} \le (1-\epsilon)\log n) < \infty$ and, by Borel-Cantelli, $P(X_{n:n} \le (1-\epsilon)\log n \text{ i.o.}) = 0$. This implies that

$$\liminf_{n\to\infty} \frac{X_{n:n}}{\log n} \ge 1-\epsilon \qquad a.s.$$

for every $\epsilon > 0$, and hence

$$\liminf_{n\to\infty} \frac{X_{n:n}}{\log n} \ge 1 \qquad a.s.$$

Combining (3) and (4) completes the proof of (b).

3. (Monte-Carlo estimation). Let $h : [0,1] \to [0,1]$ be continuous.
(i) Let $X_k = 1\{h(\xi_k) \ge \Theta_k\}$ where $\xi_1, \xi_2, \ldots$ and $\Theta_1, \Theta_2, \ldots$ are i.i.d. Uniform$(0,1)$ rv's. Show that $\overline{X}_n \to_{a.s.} \int_0^1 h(t)dt$
(ii) Let $Y_k \equiv h(\xi_k)$. Show that $\overline{Y}_n \to_{a.s.} \int_0^1 h(t)dt$.
(iii) Evaluate $Var(\overline{X}_n)$ and $Var(\overline{Y}_n)$.

Solution: (i) Now the X_k's are i.i.d. with

$$\begin{aligned} E(X_k) &= E1\{h(\xi_k) \ge \Theta_k\} = \int_0^1 \int_0^1 1\{h(t) \ge s\} ds dt \\ &= \int_0^1 \int_0^{h(t)} ds dt = \int_0^1 h(t) dt\,, \end{aligned}$$

and hence $\overline{X}_n \to_{a.s.} \int_0^1 h(t)dt$ by the SLLN.
(ii) Similarly the Y_k's are i.i.d. with $E(Y_k) = \int_0^1 h(t)dt$, so $\overline{Y}_n \to_{a.s.} \int_0^1 h(t)dt$ by the SLLN.
(iii) $Var(\overline{X}_n) = n^{-1} Var(X_1)$ and $Var(\overline{Y}_n) = n^{-1} Var(Y_1)$ where

$$E(X_1^2) = \int_0^1 \int_0^1 1\{h(t) \ge s\}^2 ds dt = \int_0^1 h(t) dt$$

and

$$E(Y_1^2) = \int_0^1 h^2(t) dt \le \int_0^1 h(t) dt$$

(since $h(t) \in [0,1]$ for all $t \in [0,1]$) so that

$$Var(Y_1) = \int_0^1 h(t)dt - \left(\int_0^1 h(t)dt\right)^2 = \int_0^1 h(t)dt \left(1 - \int_0^1 h(t)dt\right)$$

is greater than or equal to

$$Var(X_1) = \int_0^1 h^2(t)dt - \left(\int_0^1 h(t)dt\right)^2$$

with equality if and only if $\lambda(\{x : h(x) = 0 \text{ or } \ 1\}) = 1$.

4. *An investment problem.* Suppose that at the beginning of each year you can buy bonds for \$1 that are worth \$a at the end of the year or stocks that are worth a random amount $V \geq 0$. If you always invest a fixed proportion p of your wealth in bonds, then your wealth at the end of year $n+1$ is $W_{n+1} = (ap + (1-p)V_n)W_n$. Suppose that $V, V_1, V_2, \ldots$ are i.i.d. with $EV < \infty$ and $EV^{-2} < \infty$, and that $W_0 = 1$.
(i) Show that $n^{-1} \log W_n \rightarrow_{a.s.} c(p)$.
(ii) Show that the limit $c(p)$ is a concave function of p. By computing $c'(0)$ and $c'(1)$, give conditions on V that guarantee that the optimal choice of p is in $(0, 1)$.
(iii) Suppose that $P(V = 1) = P(V = 4) = 1/2$. Find the optimal p as a function of a.

Solution: (i) Now

$$W_n = \prod_{j=1}^{n-1} [ap + (1-p)V_j] W_1$$

and hence

$$n^{-1} \log W_n = \frac{1}{n} \sum_{j-1}^{n-1} \log[ap + (1-p)V_j] + n^{-1} \log W_1 \equiv \frac{1}{n} \sum_{j-1}^{n-1} X_j + n^{-1} \log W_1$$

where $X_j \equiv \log(ap + (1-p)V_j)$ are i.i.d. with

$$\begin{aligned} EX_1^+ &= E\left(\{\log(ap + (1-p)V_1)\} 1\{ap + (1-p)V_1 \geq 1\}\right) \\ &\leq E\left(\{(ap + (1-p)V_1)\} 1\{ap + (1-p)V_1 \geq 1\}\right) \\ &\leq E\{(ap + (1-p)V_1)\} = ap + (1-p)E(V_1) < \infty \end{aligned}$$

since $\log(x) \leq x - 1 \leq x$ and $E(V_1) < \infty$. Also

$$\begin{aligned} EX_1^- &= E\left(\{|\log(ap + (1-p)V_1)|\} 1\{ap + (1-p)V_1 < 1\}\right) \\ &\leq E\left([(ap + (1-p)V_1)]^{-2} 1\{ap + (1-p)V_1 < 1\}\right) \\ &\leq E\left([(ap + (1-p)V_1)]^{-2}\right) \leq (1-p)^{-2} EV_1^{-2} < \infty \end{aligned}$$

since $|\log(x)| \le x^{-2}$ for $0 < x \le 1$ and $EV_1^{-2} < \infty$. Therefore $E|X_1| = EX_1^+ + EX_1^- < \infty$, and by the SLLN

$$\begin{aligned} n^{-1}\log W_n &= \frac{1}{n}\sum_{j-1}^{n-1} X_j + n^{-1}\log W_1 \\ &\rightarrow_{a.s.} EX_1 \equiv c(p) = E\log[ap + (1-p)V]\,. \end{aligned}$$

(ii) Set $g(p) \equiv g(p; a, v) \equiv \log(v + (a - v)p)$. Since log is concave, and the composition of a linear function and a concave function is concave, g is a concave function of p for each fixed $a > 0$, $v \ge 0$, and $p \in [0, 1]$; for any $\lambda \in [0, 1]$, $p_1, p_2 \in [0, 1]$,

$$g(\lambda p_1 + \overline{\lambda} p_2) \ge \lambda g(p_1) + \overline{\lambda} g(p_2)\,.$$

Therefore

$$\begin{aligned} c(\lambda p_1 + \overline{\lambda} p_2) &= Eg(\lambda p_1 + \overline{\lambda} p_2; a, V) \\ &\ge E\left\{\lambda g(p_1; a, V) + \overline{\lambda} g(p_2; a, V)\right\} \\ &= \lambda c(p_1) + \overline{\lambda} c(p_2)\,; \end{aligned}$$

i.e. $c(p)$ is concave. Straightforward calculation yields

$$c'(p) = E\left(\frac{a - V}{ap + (1-p)V}\right)\,.$$

If $c'(0) > 0$ and $c'(1) < 0$, then the maximizer of the function $c(p)$ is in $(0, 1)$. Taking limits in the above expression for $c'(p)$ yields $c'(1) = E(1 - V/a) < 0$ if $E(V) > a$, and $c'(0) = E(a/V) - 1 > 0$ if $E(1/V) > 1/a$, or if the *harmonic mean of* V, $1/E(1/V) < a$. Thus for $1/E(1/V) < a < E(V)$, the maximizer $p_{max} \in (0, 1)$.
(iii) In the case that $P(V = 1) = P(V = 4) = 1/2$, $1/E(1/V) = 8/5$ and $EV = 5/2$. Direct computation of $c'(p)$ shows that this is zero for

$$p_{max} \equiv= p_{max}(a) = \frac{5a - 8}{2(a-1)(4-a)} \in (0, 1)$$

if $8/5 < a < 5/2$. When $a \ge 5/2$, $p_{max}(5/2) = 1$, and the maximum occurs at the boundary. When $a \le 8/5$, $p_{max} = 0$, and the maximum again occurs at the boundary. (See the attached plots.)

5. (Inversion of Laplace transforms.) Let P be a probability measure on the Borel subsets of $[0,\infty)$, and define its *Laplace transform* by $\varphi(t) = \int_0^\infty e^{-tx} dP(x)$ for $t \in [0,\infty)$. Widder's inversion formula for P from φ is:

$$\lim_{n\to\infty} \sum_{k=0}^{[nz]} \frac{(-1)^k}{k!} n^k \varphi^{(k)}(n) = P([0,z)) \tag{4}$$

for $z \in [0,\infty)$ with $P(\{z\}) = 0$. Show that (4) holds via the following steps:

(a) Differentiation of the integral k times shows that

$$\varphi^{(k)}(t) = \int_0^\infty (-x)^k e^{-tx} dP(x)\,.$$

(b) Setting $t = n$, letting $z > 0$, multiplying across by $(-1)^k n^k/k!$, and summing on k yields

$$\sum_{k=0}^{[nz]} \frac{(-1)^k}{k!} n^k \varphi^{(k)}(n) = \int_0^\infty \sum_{k=0}^{[nz]} e^{-nx} \frac{(nx)^k}{k!} dP(x)\,. \tag{5}$$

where $e^{-nx}\frac{(nx)^k}{k!} = P(S_n = k)$ and $S_n = Y_1 + \ldots + Y_n$ where $Y_1, Y_2, \ldots$ are i.i.d. Poisson(x).

(c) Use the weak law of large numbers and (5) to show that (4) holds,

Solution: (a) and (b) are self-explanatory and follow immediately. It remains only to show that the limit in (c) holds. Now as noted $e^{-nx}\frac{(nx)^k}{k!} = P(S_n = k)$ and $S_n = Y_1 + \ldots + Y_n$ where $Y_1, Y_2, \ldots$ are i.i.d. Poisson(x). Hence

$$\begin{aligned} \sum_{k=0}^{[nz]} e^{-nx} \frac{(nx)^k}{k!} &= P(S_n \le [nz]) = P(n^{-1}S_n \le n^{-1}[nz]) \\ &= P(n^{-1}S_n - n^{-1}[nz] + z \le z) \\ &= E1\{Y_n \le z\} = Eh_z(Y_n) \end{aligned}$$

where the function $h_z(y) \equiv 1\{y \le z\}$ is bounded and continuous except at the point $y = z$, and where, by the SLLN, $n^{-1}S_n \to_{a.s.} E(Y_1) = x$

and $n^{-1}[nz] \to z$ so that $Y_n \equiv n^{-1}S_n + z - n^{-1}[nz] \to_{a.s.} x$ and hence also $Y_n \to_d x$. By the Helly-Bray Theorem 5.1 it follows that

$$\sum_{k=0}^{[nz]} e^{-nx}\frac{(nx)^k}{k!} = Eh_z(Y_n) \to h_z(x) = 1\{x \le z\}$$

for $x \ne z$. Hence for $z \in [0, \infty)$ with $P(\{z\}) = 0$ we have

$$\begin{aligned}
\sum_{k=0}^{[nz]} \frac{(-1)^k}{k!} n^k \varphi^{(k)}(n) &= \int_0^\infty \sum_{k=0}^{[nz]} e^{-nx}\frac{(nx)^k}{k!} dP(x) \\
&= \int_0^\infty Eh_z(Y_n) dP(x) \\
&\to \int_0^\infty h_z(x) dP(x) \\
&= \int_0^\infty 1\{x \le z\} dP(x)
\end{aligned}$$