Statistics 522, Problem Set 1 Solutions

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1. Complete the proof of (6) in Inequality 3.4, page 181, PfS, i.e. show how the two one-sided arguments combine to yield the inequality with absolute value signs.

   Solution: Let $\tau \equiv \inf\{k \le n : |S_k| \ge \lambda\}$. Thus $[\tau = k] = [|S_1| < \lambda, \ldots |S_{k-1}| < \lambda, |S_k| \ge \lambda]$, $k = 1, \ldots, n$, and $\sum_{k=1}^n [\tau = k] = [\max_{1\le k\le n} |S_k| \ge \lambda]$. Thus, with

$$a \equiv \min_{1\le k\le n} P(|S_n - S_k| < (1-c)\lambda)\,,$$

$$\begin{aligned} aP(\max_{1\le k\le n} |S_k| \ge \lambda) &\le \sum_{k=1}^n P(|S_n - S_k| < (1-c)\lambda)P(\tau = k) \\ &= \sum_{k=1}^n P([\tau = k] \cap [|S_n - S_k| < (1-c)\lambda]) \qquad \text{by independence} \\ &= \sum_{k=1}^n P([\tau = k] \cap [|S_k| \ge \lambda] \cap [|S_n - S_k| < (1-c)\lambda]) \\ &\le \sum_{k=1}^n P([\tau = k] \cap [|S_n| \ge c\lambda]) \\ &= P(|S_n| \ge c\lambda)\,. \end{aligned}$$

2. Let $X_1, \ldots, X_n$ be independent with 0 means, and independent of the i.i.d. Rademacher rv's $\epsilon_1, \ldots, \epsilon_n$; thus $P(\epsilon_k = \pm 1) = 1/2$. Let Φ be convex and $\nearrow$ on R. Then

$$E\Phi(|\sum_1^n \epsilon_k X_k|/2) \le E\Phi(|\sum_1^n X_k|) \le E\Phi(2|\sum_1^n \epsilon_k X_k|)\,.$$

Solution: Let $X_1', \ldots, X_n'$ be an independent copy of $X_1, \ldots, X_n$. Then since $E(X_k') = 0$, it follows that

$$
\begin{aligned}
E\Phi(|\sum_1^n X_k|) &= E\Phi(|\sum_1^n (X_k - EX_k')|) \\
&\leq E_X\Phi(E_{X'}|\sum_1^n (X_k - X_k')|) \\
&\qquad \text{by Jensens' inequality since } |\cdot| \text{ is convex} \\
&\leq E_X E_{X'}\Phi(|\sum_1^n (X_k - X_k')|) \qquad \text{since } \Phi \text{ is convex} \\
&= E_{X,X',\epsilon}\Phi(|\sum_1^n \epsilon_k(X_k - X_k')|) \\
&\leq E_{X,X',\epsilon}\Phi(2\{|\sum_1^n \epsilon_k X_k| + |\sum_1^n \epsilon_k X_k'|\}/2) \\
&\qquad \text{since } \Phi \text{ is } \nearrow \\
&\leq \frac{1}{2}\left\{E\Phi(2\{|\sum_1^n \epsilon_k X_k|) + E\Phi(2\{|\sum_1^n \epsilon_k X_k'|)\right\} \\
&\qquad \text{since } \Phi \text{ is convex} \\
&= E\Phi(2\{|\sum_1^n \epsilon_k X_k|) \\
&\qquad \text{since the two terms have the same expectation.}
\end{aligned}
$$

Thus the second inequality holds (the one on the right). Similarly, to prove the first inequality (the one on the left),

$$
\begin{aligned}
E\Phi(|\sum_1^n \epsilon_k X_k|/2) &= E\Phi(|\sum_1^n \epsilon_k(X_k - EX_k')|/2) \\
&\leq E_{X,\epsilon}\Phi(E_{X'}|\sum_1^n \epsilon_k(X_k - X_k')|/2) \\
&\qquad \text{by Jensens' inequality since } |\cdot| \text{ is convex}
\end{aligned}
$$

$$
\begin{aligned}
&\le E_{X,\epsilon} E_{X'} \Phi(|\sum_1^n \epsilon_k (X_k - X_k')|/2) \qquad \text{since } \Phi \text{ is convex} \\
&= E_{X,X'} \Phi(|\sum_1^n (X_k - X_k')|/2) \\
&\le E_{X,X'} \Phi(\{|\sum_1^n X_k| + |\sum_1^n X_k'|\}/2) \\
&\qquad \text{since } \Phi \text{ is } \nearrow \\
&\le \frac{1}{2}\left\{ E\Phi(\{|\sum_1^n X_k|) + E\Phi(\{|\sum_1^n X_k'|) \right\} \\
&\qquad \text{since } \Phi \text{ is convex} \\
&= E\Phi(\{|\sum_1^n \epsilon_k X_k|) \\
&\qquad \text{since the two terms have the same expectation.}
\end{aligned}
$$

3. Suppose that X is a random variable, and $X' =_d X$ is independent of X, so that $X - X'$ is symmetric. Show that

$$P(|X| \le a) P(|X'| > t + a) \le P(|X - X'| > t),$$

and hence, by choosing a so that $P(|X| \le a) > 1/2$,

$$P(|X| > t + a) \le 2P(|X - X'| > t) \le 4P(|X| > t/2).$$

Use this last display to show that $E|X - X'|^r < \infty$ if and only if $E|X|^r < \infty$.

Solution: Since X and X' are independent,

$$P(|X| \le a)P(|X'| > t+a) = P([|X| \le a] \cap [|X'| > t+a]) = P(|X-X'| > t)$$

since $|X'| = |X' - X + X| \le |X' - X| + |X|$ implies $|X' - X| \ge |X'| - |X| > t$ on $[|X| \le a] \cap [|X'| > t + a]$. If a is chosen so that $P(|X| \le a) > 1/2$, this yields

$$P(|X| > t + a) \le 2P(|X - X'| > t) \le 4P(|X| > t/2).$$

Thus it follows that

$$\begin{aligned} E|X-X'|^r &= \int_0^\infty rt^{r-1}P(|X-X'|>t)dt \\ &\le \int_0^\infty rt^{r-1}2P(|X|>t/2)dt \\ &\le 2^{r+1}\int_0^\infty rs^{r-1}P(|X|>s)ds \\ &= 2^{r+1}E|X|^r\,. \end{aligned}$$

Alternatively, by the C_r−inequality,

$$E|X-X'|^r \le C_r\{E|X|^r+E|X'|^r\} = 2E|X|^r1\{r\le 1\}+2^rE|X|^r1\{r>1\}\,.$$

On the other hand

$$\begin{aligned} E|X|^r &= \int_0^\infty rt^{r-1}P(|X|>t)dt \\ &= \left(\int_0^a + \int_a^\infty\right) rt^{r-1}P(|X|>t)dt \\ &\le a^r + \int_a^\infty rt^{r-1}P(|X|>t)dt \\ &= a^r + \int_0^\infty r(s+a)^{r-1}P(|X|>s+a)ds \\ &= a^r + \left(\int_0^a + \int_a^\infty\right) r(s+a)^{r-1}P(|X|>s+a)ds \\ &\le a^r + (2a)^r - a^r + \int_a^\infty rs^{r-1}(1+\frac{a}{s})^{r-1}2P(|X-X'|>s)ds \\ &\le (2a)^r + 2^r\int_a^\infty rs^{r-1}P(|X-X'|>s)ds \\ &\le (2a)^r + 2^rE|X-X'|^r\,. \end{aligned}$$

It follows from the two inequalities that $E|X-X'|^r<\infty$ if and only if $E|X|^r<\infty$.

4. (A weak law of large numbers under the assumption of uncorrelated summands.) Suppose that $X_1, X_2, \ldots$ are uncorrelated and $E(X_j^2) \le$

$M < \infty$ for all $j \geq 1$. Show that $\overline{X} - E(\overline{X}_n) = (S_n - ES_n)/n \rightarrow_p 0$ and $\overline{X}_n - E(\overline{X}_n) \rightarrow_2 0$ as $n \rightarrow \infty$.

Solution: Since the X_n's are uncorrelated,

$$Var(S_n) = \sum_{j=1}^{n} Var(X_j) + \sum_{i \neq j} Cov(X_i, X_j) = \sum_{j=1}^{n} Var(X_j) \leq \sum_{j=1}^{n} E(X_j^2) \leq nM$$

and hence $Var(\overline{X}_n) = n^{-2} Var(S_n) \leq M/n$. Therefore, by Chebychev's inequality

$$P(|\overline{X}_n - E(\overline{X}_n)| \geq \epsilon) \leq \epsilon^{-2} Var(\overline{X}_n \leq \epsilon^{-2} \frac{M}{n} \rightarrow 0$$

for every $\epsilon > 0$; i.e. $\overline{X} - E(\overline{X}_n) \rightarrow_p 0$. Since $E[\overline{X}_n - E(\overline{X}_n)]^2 = Var(\overline{X}_n) \leq M/n \rightarrow 0$, we also have $\overline{X} - E(\overline{X}_n) \rightarrow_2 0$.

5. (L_1-convergence in the SLLN.) Suppose that $X_1, \ldots, X_n, \ldots$ are i.i.d with $E|X_1| < \infty$. Show that $\overline{X}_n \rightarrow_1 \mu \equiv E(X_1)$; i.e. $E|\overline{X}_n - \mu| \rightarrow 0$ as $n \rightarrow \infty$. [Hint: show that $|\overline{X}_n| \leq Y_n$ where Y_n is uniformly integrable, and that this implies the uniform integrability of $\overline{X}_n$.

   Solution: Note that

$$|\overline{X}_n| \leq n^{-1} \sum_{i=1}^{n} |X_i| \equiv Y_n$$

   where Y_n is uniformly integrable by Vitali's theorem: $E(Y_n) = E|X_1|$ for every n and $Y_n \rightarrow_{a.s.} E|X_1| \equiv Y_0$ by the strong law of large numbers and $E(Y_n) \rightarrow E(Y_0) = E|X_1|$. It follows that $\{\overline{X}_n\}$ is uniformly integrable, and by Vitali's theorem again $E|\overline{X}_n - \mu| \rightarrow 0$ as $n \rightarrow \infty$; i.e. $\overline{X}_n \rightarrow_1 \mu \equiv E(X_1)$.