Statistics 522, Final Exam

Wellner; 3/5/99

1. (36 points)
A. Give an example of a martingale with $X_n \to_p 0$, but $X_n \not\to_{a.s.} 0$. [Hint: Set $X_0 = 0$, and define $X_1, \ldots$ as follows: $P(X_k = \pm 1 | X_{k-1} = 0) = 1/(2k)$, $P(X_k = 0 | X_{k-1} = 0) = 1 - 1/k$; $P(X_k = kX_{k-1} | X_{k-1}) = (1/k)1_{[X_{k-1} \neq 0]}$, $P(X_k = 0 | X_{k-1}) = (1 - 1/k)1_{[X_{k-1} \neq 0]}$.]
B. Give an example of a martingale $\{X_n, \mathcal{A}_n\}$ with $X_n \to_{a.s.} -\infty$. [Hint: let $X_n = Y_1 + \cdots + Y_n$ where Y_i are independent with $E(Y_i) = 0$ and $P(Y_i = -1) = 1 - \epsilon_i$, then find an appropriate choice of the ϵ_i.]

Solution: A. Note that X_k takes on integer values, and

$$\begin{aligned} P(X_k \neq 0) &= E\,P(X_k \neq 0 | X_{k-1}) \\ &= E\left(\frac{2}{2k}1_{[X_{k-1}=0]} + \frac{1}{k}1_{[X_{k-1}\neq 0]}\right) \\ &= \frac{1}{k}P(X_k = 0) + \frac{1}{k}P(X_{k-1} \neq 0) \\ &= \frac{1}{k}(1 - P(X_k \neq 0) + \frac{1}{k}P(X_{k-1} \neq 0) \\ &= \frac{1}{k} \to 0\,, \end{aligned}$$

and hence $P(|X_k| > \epsilon) \le 1/k \to 0$ for any $0 < \epsilon < 1$. Thus $X_k \to_p 0$. To show that $X_n \not\to_{a.s.} 0$, we will show that $P([|X_n| \ge \epsilon\ i.o.]^c) = 0$ for each $\epsilon \in (0,1)$, and hence $P(|X_n| \ge \epsilon\ i.o.) = 1$. But

$$\begin{aligned} P([|X_n| \ge \epsilon\ i.o.]^c) &= P(\cup_{n=1}^{\infty} \cap_{m=n}^{\infty} [X_m = 0]) \\ &= \lim_{n\to\infty} P(\cap_{m=n}^{\infty}[X_m = 0]) \\ &= \lim_{n\to\infty} P(X_n = 0,\, X_{n+1} = 0, \ldots)\,. \end{aligned}$$

But for each fixed n

$$P(X_n = 0,\, X_{n+1} = 0, \ldots) = \lim_{N\to\infty} P(X_n = 0, \ldots, X_N = 0)$$

$$
\begin{aligned}
&= (1-\frac{1}{N})(1-\frac{1}{N-1})\cdots(1-\frac{1}{n}) \\
&= \frac{N-1}{N}\frac{N-2}{N-1}\cdots\frac{n-1}{n} \\
&= \frac{n-1}{N} \to 0\,,
\end{aligned}
$$

and hence the above limit is 0 and $P(|X_n| \ge \epsilon\ i.o.) = 1$. Thus $X_n \not\to_{a.s.} 0$.

B. If $Y_1, Y_2, \ldots$ are independent with $P(Y_i = -1) = 1 - \epsilon_i$, $P(Y_i = y_i) = \epsilon_i \to 0$ with $y_i \equiv (1-\epsilon_i)/\epsilon_i$, then $E(Y_i) = 0$. Thus an easy calculation shows that $\{X_n, \mathcal{A}_n\}$ is a mean zero martingale. Moreover, $P(Y_i = y_i \quad i.o.) = 0$ if $\sum_1^\infty \epsilon_i < \infty$; e.g. take $\epsilon_i = i^{-2}$. Then we have $P([Y_i = y_i\ i.o.]^c) = P(Y_i = -1 \quad a.a.) = 1$, and this implies that $X_n = Y_1 + \cdots + Y_n \to -\infty$ a.s..

2. (48 points).
A. Prove the following:
Lemma: If $\sigma-$fields $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, a $\sigma-$field, and $Y_n \to_1 Y$, then $E(Y_n|\mathcal{F}_n) \to_1 E(Y|\mathcal{F}_\infty)$.
B. Prove the following:
Lemma: Suppose that $\sigma-$fields $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, a $\sigma-$field, $Y_n \to_{a.s.} Y$, and $|Y_n| \le Z$ where $Z \in L_1$. Show that $E(Y_n|\mathcal{F}_n) \to_{a.s.} E(Y|\mathcal{F}_\infty)$.
C. If $\{X_n\}$ is uniformly integrable and $X_n \to_{a.s.} X$, then $X_n \to_1 X$, and A above shows that $E(X_n|\mathcal{F}) \to_1 E(X|\mathcal{F})$. Find an example to show that $E(X_n|\mathcal{F})$ need not converge a.s.
[Hint: Let $Y_1, Y_2, \ldots$ and $Z_1, Z_2, \ldots$ be independent rv's with $P(Y_n = 1) = 1/n$, $P(Y_n = 0) = 1 - 1/n$, $P(Z_n = n) = 1/n$, $P(Z_n = 0) = 1 - 1/n$. Consider $X_n \equiv Y_n Z_n$ and $\mathcal{F} = \sigma[Y_1, Y_2, \ldots]$.]

Solution: A. Note that since $Y_n \to_1 Y$, $Y \in L_1$, and

$$
\begin{aligned}
E|E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)| &\le E|E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_n)| \\
&\qquad + E|E(Y|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)| \\
&\le E\{E(|Y_n - Y||\mathcal{F}_n)\} + E|E(Y|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)| \\
&= E(|Y_n - Y|) + E|E(Y|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)|
\end{aligned}
$$

where the first term converges to zero since $Y_n \to_1 Y$, and the second term converges to zero since $Z_n \equiv E(Y|\mathcal{F}_n)$ is a Doob-martingale, and hence converges a.s. and in L_1.
B. Now write

$$\begin{aligned} |E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)| &\leq |E(Y_n|\mathcal{F}_n) - E(Y_n|\mathcal{F}_\infty)| + |E(Y_n|\mathcal{F}_\infty) - E(Y|\mathcal{F}_\infty)| \\ &\leq |E(|Y_n - Y||\mathcal{F}_n) + |E(Y_n - Y|\mathcal{F}_\infty)|\,. \end{aligned}$$

Here the second term converges a.s. surely to zero by the dominated convergence theorem for conditional expectations using the dominating function $|Y_n - Y| \leq 2Z$ for all n. For any fixed $m \leq n$, the first term is bounded above by

$$\begin{aligned} E(\sup_{k\geq m} |Y_k - Y||\mathcal{F}_n) &\to_{a.s.} E(\sup_{k\geq m} |Y_k - Y||\mathcal{F}_\infty) \quad \text{as } n \to \infty \\ &\searrow_{a.s.} E(0|\mathcal{F}_\infty) = 0 \quad \text{as } m \to \infty \end{aligned}$$

using first the a.s. convergence of the L_1- bounded Doob martingale $\{E(\sup_{k\geq m} |Y_k - Y||\mathcal{F}_n), \mathcal{F}_n\}$, for each fixed m, and then the dominated convergence theorem for conditional expectations (with the dominating function $2Z$ again).
C. With Y_n and Z_n as described in the hint, $P(|X_n| > \epsilon) = P(Y_n = 1, Z_n = n) = 1/n^2$, and hence by Borel-Cantelli, $P(|X_n| > \epsilon \;\; i.o.) = 0$, and hence $X_n \to_{a.s.} 0 \equiv X$. Furthermore, $E|X_n - X| = EX_n = E(Y_n)E(Z_n) = (1/n) \cdot 1 \to 0$, so $X_n \to_1 0 = X$, and $\{X_n\}$ is uniformly integrable by Vitali's theorem. However,

$$E(X_n|\mathcal{F}) = E(X_n|Y_n) = E(Y_n Z_n|Y_n) = Y_n E(Z_n) = Y_n$$

since $E(Z_n) = 1$. Where $P(|Y_n| > \epsilon) = 1/n$ for $\epsilon \in (0,1)$, and hence by the second Borel-Cantelli lemma, $P(|Y_n| > \epsilon \;\; i.o.) = 1$; i.e. $Y_n \not\to_{a.s.} 0$. Thus $E(X_n|\mathcal{F}) = Y_n \not\to_{a.s} 0$.

3. (32 points).
Suppose that $Z_1, Z_2, \ldots$ are i.i.d. $N(0,1)$ rv's. Let $S_n \equiv \sum_{k=1}^n Z_k$, and define

$$Y_n \equiv \exp(aS_n - bn)\,.$$

A. Prove that $Y_n \to_{a.s.} 0$ if and only if $b > 0$.
B. For $r \geq 1$, prove that $Y_n \to_r 0$ if and only if $r < 2b/a^2$.

Solution: A. To prove that $Y_n \to_{a.s.} 0$ if and only if $b > 0$, first suppose that $b > 0$. Then, by the SLLN,

$$a\frac{S_n}{n} - b \to_{a.s.} a \cdot 0 - b < 0\,,$$

so that $aS_n - bn \to_{a.s.} -\infty$, and hence by the continuous mapping theorem, $Y_n = \exp(aS_n - bn) \to_{a.s.} 0$.
Now suppose that $Y_n \to_{a.s.} 0$. Then we have $\log(Y_n) = aS_n - bn \to_{a.s.} -\infty$; i.e. for any large $M > 0$, there is an $N = N_M(\omega)$ such that for all ω in a set A with probability 1, there is an $N = N_M(\omega)$ so that for $n \geq N$ we have

$$aS_n(\omega) - bn \leq -M \quad \text{for} \qquad n \geq N_M(\omega)\,;$$

i.e.

$$a\frac{S_n(\omega)}{n} - b \leq -\frac{M}{n} < 0, \quad n \geq N_M(\omega)\,.$$

Since we know that $S_n/n \to_{a.s.} 0$, with probability one we can also make $aS_n(\omega)/n$ arbitrarily small for $n >$some $N'(\omega)$. Combining these two facts, it is clear that the above inequality can hold only $-b < 0$; i.e. $b > 0$. (Alternatively: since $S_n \sim N(0, n)$, if $b \leq 0$ it follows that

$$P(Y_n \geq 1) = P(aS_n - bn \geq 0) = P(aS_n \geq bn) \geq P(S_n \geq 0) = 1/2\,,$$

and hence $Y_n \not\to_{a.s.} 0$.)
B. Now write $Y_n = \prod_{i=1}^n X_i$ where $X_i \equiv e^{aZ_i - b}$ are i.i.d. with

$$EX_i^r = E\exp(raX_i)\exp(-rb) = e^{r^2a^2/2 - rb} \equiv \mu_r\,.$$

Note that $\mu_r < 1$ if and only if $r < 2b/a^2$. Since $Y_n^r = \prod_{i=1}^n X_i^r$, it follows by from the X_i's being i.i.d. that

$$EY_n^r = \{\mu_r\}^n \to 0$$

if and only if $r < 2b/a^2$.

Do **either** problem 4 **or** problem 5.

4. (42 points).
Suppose that $X_1, X_2, \ldots$ are i.i.d non-negative random variables with $E(X_1) > 1$. Let $M_n \equiv \prod_{j=1}^n X_j$. Let $\mathcal{A}_n \equiv \sigma[M_1, \ldots, M_n]$.
A. Show that $\{M_n, \mathcal{A}_n\}$ is a sub-martingale.
B. Find the Doob decomposition of $\{M_n, \mathcal{A}_n\}$.
C. If $P(X_1 = 0) > 0$, show that $M_n \rightarrow_{a.s.} 0$ although $\{M_n\}$ is not integrable (i.e. L_1 bounded).

Solution: A. Now

$$\begin{aligned} E(M_{n+1}|\mathcal{A}_n) &= E(M_n X_{n+1}|\mathcal{A}_n) \\ &=_{a.s.} M_n E(X_{n+1}|\mathcal{A}_n) \\ &=_{a.s.} M_n E(X_{n+1}) \\ &= \mu M_n > M_n\,, \end{aligned}$$

so $\{M_n, \mathcal{A}_n\}$ is a sub-martingale.
B. Now with $M_0 \equiv 1$,

$$\begin{aligned} A_n &= \sum_{j=1}^n \{E(M_j|\mathcal{A}_{j-1}) - M_{j-1}\} + EM_0 \\ &= \sum_{j=1}^n (\mu M_{j-1} - M_{j-1}) + 1 \\ &= (\mu - 1)\sum_{j=1}^n M_j + 1\,. \end{aligned}$$

Note that this yields

$$\begin{aligned} E(A_n) &= (\mu - 1)\sum_{j=1}^n \mu^{j-1} + 1 \\ &= (\mu - 1)(1 + \mu + \cdots + \mu^{n-1}) + 1 \\ &= \mu^n - 1 + 1 = \mu^n = E(M_n)\,. \end{aligned}$$

C. Now $E(M_n) = \mu^n \rightarrow \infty$ since $\mu > 1$, so that $\{M_n\}$ is not L_1 bounded; i.e. $\{M_n\}$ is not *integrable*. But if $P(X_1 = 0) > 0$, then $\sum_i P(X_i = 0) = \infty$, and hence by the second Borel-Cantelli lemma, $P([X_n = 0]\ i.o.) = 1$, and hence $M_n \rightarrow_{a.s.} 0$.

5. (42 points).
Let $X_1, X_2, \ldots$ be independent random variables with means $E(X_j) = 0$ and finite variances σ_j^2, $j = 1, 2, \ldots$. Let $S_n = X_1 + \ldots + X_n$, and $\mathcal{A}_n \equiv \sigma[X_1, \ldots, X_n]$, $n = 1, 2, \ldots$.
A. Show that $\{S_n^2, \mathcal{A}_n\}$ is a submartingale.
B. Evaluate the Doob decomposition of $\{S_n^2\}$.
C. If $\sum_{j=1}^{\infty} \sigma_j^2 < \infty$, show that the martingale $\{S_n\}$ converges a.s. (giving a martingale proof of part of the three-series theorem).

Solution: $\{S_n^2, \mathcal{A}_n\}$ is a sub-martingale since S_n is a mean 0 martingale, and hence, by the conditional verion of Jensen's inequality,

$$E(S_{n+1}^s | \mathcal{A}_n) \ge \{E(S_{n+1}|\mathcal{A}_n)\}^2 = S_n^2 \qquad a.s.$$

B. In this case,

$$S_n^2 = (S_n^2 - \langle S_n \rangle) + \langle S_n \rangle$$

where the predictable variation process is $\langle S_n \rangle = \sum_{j=1}^{n} \sigma_j^2$.
C. Now

$$E(S_n^2) = E\langle S_n \rangle = \sum_{j=1}^{n} \sigma_j^2 \nearrow \sum_{j=1}^{\infty} \sigma_j^2 < \infty\,.$$

So the sub-martingale S_n^2 converges a.s. by the m.g. convergence theorem; i.e. $S_n \to_{a.s.} \sum_{j=1}^{\infty} X_j$.

6. (40 points).
Let $X_1, \ldots, X_n$ be i.i.d. Poisson(1) random variables, set $S_n = X_1 + \ldots + X_n$, and $Z_n \equiv (S_n - n)/\sqrt{n}$. Prove Stirling's formula, $n! \sim \sqrt{2\pi n}(n/e)^n$, by showing that each of the following steps is valid:
A.

$$E\left(\frac{S_n - n}{\sqrt{n}}\right)^{-} = e^{-n} \sum_{k=0}^{n} \frac{n-k}{\sqrt{n}} \frac{n^k}{k!} = \frac{n^{n+1/2}e^{-n}}{n!}\,.$$

B. $Z_n \to_d Z \sim N(0,1)$.
C. $EZ_n^- \to EZ^- = 1/\sqrt{2\pi}$.
D. $n! \sim \sqrt{2\pi}n^{n+1/2}e^{-n} = \sqrt{2\pi n}(n/e)^n$.

Solution: A. First, $S_n = X_1 + \cdots + X_n \sim \text{Poisson}(n)$, and hence we can compute

$$
\begin{aligned}
E\left(\frac{S_n - n}{\sqrt{n}}\right)^- &= \sum_{k=0}^{n}\left(\frac{n-k}{\sqrt{n}}\right)\frac{n^k e^{-n}}{k!} \\
&= \frac{n}{\sqrt{n}} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} - \sum_{k=1}^{n} \frac{k}{\sqrt{n}} \frac{n^k e^{-n}}{k!} \\
&= \sqrt{n} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} - \frac{e^{-n}}{\sqrt{n}} \sum_{k=1}^{n} \frac{n^k}{(k-1)!} \\
&= \sqrt{n} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} - \sqrt{n} e^{-n} \sum_{k=1}^{n} \frac{n^{k-1}}{(k-1)!} \\
&= \sqrt{n} e^{-n} \frac{n^n}{n!}.
\end{aligned}
$$

B. Since $E(X_1) = 1$, $Var(X_1) = 1$, the Lindeberg - Lévy CLT yields

$$Z_n \equiv \frac{S_n - n}{\sqrt{n}} = \sqrt{n}(\overline{X}_n - 1) \to_d N(0,1).$$

C. Note that $E(Z_n^2) = 1$ for all n, so $\{Z_n^-\}$ is uniformly integrable and it follows that

$$EZ_n^- \to EZ^- = E(Z^+) = \int_0^\infty z \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}}.$$

D. It follows from A and D that

$$\frac{n^{1/2}(n/e)^n}{n!} = E(Z_n^-) \to E(Z^-) = \frac{1}{\sqrt{2\pi}}.$$

This is equivalent to

$$\frac{\sqrt{2\pi}(n/e)^n}{n!} \to 1,$$

i.e. $n! \sim \sqrt{2\pi n}(n/e)^n$.

7. (36 points).
Suppose that $X_1, X_2, \ldots$ are i.i.d. mean 0 rv's with $E(X_1^2) < \infty$. Let

$S_n = \sum_{j=1}^n X_j$, $R_n = \sum_{j=1}^n S_j$, and $\mathcal{A}_n = \sigma[X_1, \ldots, X_n]$.
A. Show with $Y_n \equiv R_n - nS_n$, $\{Y_n, \mathcal{A}_n\}$ is a martingale.
B. Find the Doob-Meyer decomposition of the submartingale Y_n^2, and use it to compute $E(Y_n^2)$.

Solution: A. First, note that summation by parts yields

$$\begin{aligned} R_n &= \sum_{j=1}^n S_j = \sum_{j=1}^n \sum_{i=1}^j X_i \\ &= \sum_{i=1}^n X_i \sum_{j=1}^n 1_{[j \geq i]} \\ &= \sum_{i=1}^n X_i (n - i + 1) \\ &= nS_n - \sum_{i=1}^N (i-1) X_i \,. \end{aligned}$$

It follows that

$$Y_n \equiv R_n - nS_n = -\sum_{i=1}^n (i-1) X_i$$

and hence, by independence of the X_i's,

$$\begin{aligned} E(Y_{n+1} | \mathcal{A}_n) &= Y_n - E(nX_{n+1} | \mathcal{A}_n) \\ &= Y_n - nE(X_{n+1}) = Y_n \qquad a.s.\,, \end{aligned}$$

i.e. $\{Y_n, \mathcal{A}_n\}$ is a martingale.
B. Now $Y_n^2 - \langle Y_n \rangle$ is a martingale where

$$\langle Y_n \rangle = \sum_{i=1}^n (i-1)^2 \sigma^2 = \sigma^2 \sum_{k=1}^{n-1} k^2 = \sigma^2 \frac{(n-1)n(2n-1)}{6} \,.$$

Thus it follows that

$$E(Y_n^2) = E\langle Y_n \rangle = \sigma^2 \sum_{k=1}^{n-1} k^2 \,.$$