Statistics 522, Midterm Exam Solutions

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1. (18 points). **Define** *three* of the following four terms:
   (a) The conditional expectation of a random variable X given a (sub-) sigma-field $\mathcal{D}$.
   (b) A martingale, sub-martingale, and super-martingale.
   (c) A stopping time (relative to a filtration $\mathcal{A}_n$).
   (d) The compensator of a sub-martingale.

   Solution: See chapters 10 and 18.

2. (27 points). Give careful **statements** of *three* of the following five theorems or results:
   (a) The S-mg convergence theorem.
   (b) The three-series theorem.
   (c) The strong law of large numbers.
   (d) The step-wise smoothing property of conditional expectations.
   (e) The interpretation of conditional expectations in terms of an (orthogonal) projection onto $L_2(\Omega, \mathcal{G}, P)$ where $\mathcal{G} \subset \mathcal{A}$.

   Solution: See chapters 10 and 18.

3. (42 points). Suppose that $X_0 = 1$, and let $X_n \sim \text{Uniform}(0, X_{n-1})$ for $n \geq 1$. Let $\mathcal{A}_n \equiv \sigma[X_0, X_1, \ldots, X_n]$ for $n = 0, 1, \ldots$.
   (a) Show that with $Y_n \equiv 2^n X_n$, $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is a martingale, and hence that $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is a non-negative super-martingale.
   (b) Apply the s-mg convergence theorem to the martingale $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$.
   (c) There is no convergence theorem stated for a non-negative supermartingale in PfS, but based on what you know about the s-martingale convergence theorem and the reversed martingale convergence theorem, state a convergence theorem for non-negative supermartingales and apply it to $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$. What is the a.s. limit of X_n in the present case?
   (d) Is there any connection between the martingale $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ and Kakutani's product martingales?

(e) Use (d) to determine whether or not the martingale $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is uniformly integrable. Does convergence hold in L_1?
(f) Compute $E(X_{n+1}^2|\mathcal{A}_n)$ and $E(Y_{n+1}^2|\mathcal{A}_n)$.
(g) Use the computation in (f) to find a martingale related to $\{X_n^2\}$, and use it to compute $E(X_n^2)$ and $E(Y_n^2)$. Are either $\{X_n\}$ or $\{Y_n\}$ square-integrable?

Solution: (a) Since $X_{n+1} \sim$Uniform$(0, X_n)$, $E(X_{n+1}|\mathcal{A}_n) = X_n/2 \leq X_n$ a.s., and hence $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is a non-negative supermartingale. Furthermore,

$$E(Y_{n+1}|\mathcal{A}_n) = 2^{n+1}E(X_{n+1}|\mathcal{A}_n) = 2^{n+1}X_n/2 = 2^n X_n = Y_n \qquad \text{a.s.}$$

so that $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is a non-negative martingale.
(b) Now $E(Y_n) = E(Y_0) = E(X_0) = 1$, for all n, and $Y_n \geq 0$, so $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is an L_1-bounded martingale, and hence by the s-martingale convergence theorem, $Y_n \to_{a.s.} Y_\infty$.
(c) **Theorem.** If $\{X_n, \mathcal{A}_n\}$ is a non-negative super-martingale, then $X_n \to_{a.s.} X_\infty$.
Proof. $W_n \equiv -X_n \leq 0$ is a sub-martingale with $EW_n^+ = 0$. Thus by the s-mg convergence theorem, $W_n = -X_n \to_{a.s.} -X_\infty \equiv W_\infty$. In the present case, $X_n = 2^{-n}Y_n \to_{a.s.} 0 \cdot Y_\infty = 0$.
(d) Yes. Since $X_{n+1} \sim$Uniform$(0, X_n) =_d X_n\cdot$Uniform$(0,1) \equiv X_n W_{n+1}$ where $W_{n+1} \sim$Uniform$(0,1)$, it follows that

$$X_n = \prod_{k=1}^{n} W_k$$

where $W_1, W_2, \ldots$ are independent Uniform$(0,1)$ rv's. Thus $Y_n = 2^n X_n = \prod_{k=1}^{n}(2W_k)$ where $2W_k \sim$Uniform$(0,2)$ are i.i.d. with $E(2W_k) = 1$. Thus Y_n is just the Kakutani product martingale formed from i.i.d. Uniform$(0,2)$ random variables.
(e) Since $a_k = E(2W_k)^{1/2} = \sqrt{2}\int_0^1 u^{1/2}du = \sqrt{2}(2/3) < 1$,

$$\prod_{k=1}^{n} a_k = \left(\sqrt{2}\frac{2}{3}\right)^n \to 0$$

as $n \to \infty$; i.e. $\prod_{k=1}^{\infty} a_k = 0$. Thus by Kakutani's theorem the martingale $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is *not uniformly integrable*, and L_1-convergence fails:

$1 = E(Y_n) \not\to E(Y_\infty) = 0$.
(f) Note that

$$\begin{aligned} E(X_{n+1}^2|\mathcal{A}_n) &= Var(X_{n+1}|\mathcal{A}_n) + (E(X_{n+1}|\mathcal{A}_n))^2 \\ &= \frac{1}{12}X_n^2 + \frac{1}{4}X_n^2 = \frac{1}{3}X_n^2 \qquad \text{a.s.} \end{aligned}$$

and therefore

$$E(Y_{n+1}^2|\mathcal{A}_n) = E(2^{2(n+1)}X_{n+1}^2|\mathcal{A}_n) = 2^{2(n+1)}\frac{1}{3}X_n^2 = \frac{4}{3}Y_n^2 .$$

(g) By the calculation in (f),

$$E((3/4)^{n+1}Y_{n+1}^2|\mathcal{A}_n) = (3/4)^{n+1}\frac{4}{3}Y_n^2 = (3/4)^n Y_n^2 \qquad \text{a.s.},$$

and therefore $Z_n \equiv (3/4)^n Y_n^2$ is a martingale. Thus $E((3/4)^n Y_n^2) = E(Z_n) = E(Z_0) = 1$, and it follows that $E(Y_n^2) = (4/3)^n$, and $E(2^{2n}X_n^2) = (4/3)^n$, or $E(X_n^2) = (1/3)^n$. Thus $\{X_n\}$ is square-integrable, but $\{Y_n\}$ is not square-integrable.

4. (26 points). Suppose that $X \in L_2(P)$ and $\mathcal{D}$ is a sub-sigma field. The conditional variance of X given $\mathcal{D}$ is defined by

$$Var(X|\mathcal{D}) = E\left\{(X - E(X|\mathcal{D}))^2|\mathcal{D}\right\} .$$

(a) Prove that

$$Var(X) = E[Var(X|\mathcal{D})] + Var(E(X|\mathcal{D})) .$$

(b) Interpret the formula in (a) geometrically.

Solution: (a) First we write

$$\begin{aligned} Var(X) &= E(X - E(X))^2 \\ &= E\left\{(X - E(X|\mathcal{D}) + E(X|\mathcal{D}) - E(X))^2\right\} \\ &= E\left\{(X - E(X|\mathcal{D}))^2 + 2(X - E(X|\mathcal{D}))(E(X|\mathcal{D}) - E(X)) \right. \\ &\qquad \left. + \ (E(X|\mathcal{D}) - E(X))^2\right\} \\ &= EE\left\{(X - E(X|\mathcal{D}))^2|\mathcal{D}\right\} + E(E(X|\mathcal{D}) - E(X))^2 \\ &= EVar(X|\mathcal{D}) + Var(E(X|\mathcal{D})) \end{aligned}$$

since, by computing conditionally on $\mathcal{D}$,

$$
\begin{aligned}
& E\{(X - E(X|\mathcal{D}))(E(X|\mathcal{D}) - E(X))\} \\
&= EE\{(X - E(X|\mathcal{D}))(E(X|\mathcal{D}) - E(X))|\mathcal{D}\} \\
&= E((E(X|\mathcal{D}) - E(X))E\{(X - E(X|\mathcal{D}))|\mathcal{D}\}) \\
&= E\{(E(X|\mathcal{D}) - E(X))(E(X|\mathcal{D}) - E(X|\mathcal{D}))\} \\
&= E\{(E(X|\mathcal{D}) - E(X)) \cdot 0\} = 0\,.
\end{aligned}
$$

(b) In the proof of the identity in (a), the argument producing the 0 for the middle term expresses the orthogonality of $X - E(X|\mathcal{D})$ to all $\mathcal{D}$-measurable functions, and hence, in particular, to $E(X|\mathcal{D}) - E(X)$. Thus the geometric picture is as follows:

5. (26 points). Suppose that $Z_1, Z_2, \ldots$ are i.i.d. $N(0,1)$ random variables. Let $\underline{Z} \equiv (Z_1, \ldots, Z_n)$, so that $\underline{Z} \sim N_n(0, I_n)$ where I_n is the $n \times n$ identity matrix, and let A be an orthogonal matrix (i.e. an $n \times n$ matrix with $A^T A = I_n$). Let $S^{n-1} \equiv \{\underline{x} \in R^n : |\underline{x}| = 1\}$.
(a) Show that $A\underline{Z} \sim N_n(0, I_n)$ for any orthogonal matrix A.
(b) Use the result of (a) to show that $\underline{Z}_n/|\underline{Z}_n| \sim \text{Uniform}(S^{n-1})$. You may use the fact that the Uniform distribution on S^{n-1} is the unique distribution which is invariant under orthogonal transformations.
(c) Show that $R_n = (Z_1^2 + \cdots + Z_n^2)^{1/2}$ satisfies $R_n/\sqrt{n} \to_{a.s.} 1$.

(d) Use (a) - (c) to show that if $\underline{Y}_n \sim$ Uniform(S^{n-1}), then for any fixed integer k it follows that $\sqrt{n}(Y_{n1}, \ldots, Y_{nk}) \to_d N_k(0, I_k)$.

Solution: (a) If A is an orthogonal matrix,

$$E(A\underline{Z}) = AE(\underline{Z}) = A\underline{0} = \underline{0}\,,$$

and

$$E(A\underline{Z}\underline{Z}^T A^T) = AE(\underline{Z}\underline{Z}^T)A^T = AI_nA^T = AA^T = I_n$$

and since linear combinations of normal variables are normal, we find that $A\underline{Z} \sim N_n(0, I_n)$.
(b) For any Borel set $B \subset S^{n-1}$ and any orthogonal matrix A we have $|A\underline{Z}|^2 = \underline{Z}^T A^T A\underline{Z} = \underline{Z}^T\underline{Z} = |\underline{Z}|^2$, and hence

$$\begin{aligned} P(\underline{Z}/|\underline{Z}| \in B) &= P(A\underline{Z}/|\underline{Z}| \in A(B)) \\ &= P(A\underline{Z}/|A\underline{Z}| \in A(B)) \\ &= P(\underline{Z}/|\underline{Z}| \in A(B)) \end{aligned}$$

where $A(B) \equiv \{A\underline{x} : \underline{x} \in B \subset S^{n-1}\}$. Thus the distribution of $\underline{Z}/|\underline{Z}|$ is invariant under orthogonal transformations; since there is only one such distribution on S^{n-1}, the uniform distribution, we conclude that $\underline{Z}/|\underline{Z}| \sim$Uniform$(S^{n-1})$.
(c) Now

$$\frac{1}{n}R_n^2 = \frac{1}{n}(Z_1^2 + \cdots + Z_n^2) \to_{a.s.} 1$$

by the SLLN since $Z_1^2, Z_2^2, \ldots$ are i.i.d. with $E(Z_i^2) = 1$. Thus $R_n/\sqrt{n} \to_{a.s.} 1$ by the continuous mapping theorem.
(d) If $\underline{Y}_n \sim$Uniform(S^{n-1}), then since $R_n = |\underline{Z}_n|$,

$$\begin{aligned} \sqrt{n}(Y_{n1}, \ldots, Y_{nk}) \quad &=_d \quad \sqrt{n}(\frac{Z_1}{|\underline{Z}|}, \ldots, \frac{Z_k}{|\underline{Z}|}) \\ &= \quad (Z_1, \ldots, Z_k)\frac{\sqrt{n}}{R_n} \\ &\to_{a.s.} \quad (Z_1, \ldots, Z_k) \end{aligned}$$

by the result of (c). Hence we conclude that

$$\sqrt{n}(Y_{n1}, \ldots, Y_{nk}) \to_d N_k(0, I_k)\,.$$

6. (26 points). Let $\xi_1, \xi_2, \ldots$ be i.i.d. Uniform$(0,1)$. Let $G(t) = t$ for $t \in [0,1]$, $G(t) = 0$ for $t \leq 0$, $G(t) = 1$ for $t \geq 1$, be the Uniform$(0,1)$ distribution function. Let $\mathbb{G}_n(t) = n^{-1} \sum_{i=1}^n 1_{[0,t]}(\xi_i)$, the uniform empirical distribution function. Prove the Glivenko-Cantelli theorem for $\mathbb{G}_n$: $\|\mathbb{G}_n - G\|_\infty \equiv \sup_{0 \leq t \leq 1} |\mathbb{G}_n(t) - t| \rightarrow_{a.s.} 0$.

Solution: See PfS section 10.5, page 190.