Statistics 522, Problem Set 2

Wellner; 1/13/99

Reading: Shorack, PfS, Chapter 10, pages 179 - 193; 200 - 213;
Kallenberg, FMP, Chapter 1, pages 1-6, and Appendix 1, pages 455-457.

Due: Wednesday, January 20, 1999.

1. Suppose that $X_1, X_2, \ldots$ are independent random variables with $P(X_n = n^2 - 1) = 1/n^2$, $P(X_n = -1) = 1 - 1/n^2$. Show that $E(X_n) = 0$ for all n (so the game is "fair" for every n), but that

$$\overline{X}_n = n^{-1} S_n = n^{-1} \sum_{i=1}^{n} X_i \rightarrow_{a.s.} -1$$

as $n \rightarrow \infty$.

2. Exercise 10.4.13, PfS, page 189. (Note that $x_{n:n}$ should be $X_{n:n} \equiv \max_{1 \le i \le n} X_i$.)

3. Exercise 10.5.1, PfS, page 192.

4. *An investment problem.* Suppose that at the beginning of each year you can buy bonds for \$1 that are worth \$a at the end of the year or stocks that are worth a random amount $V \ge 0$. If you always invest a fixed proportion p of your wealth in bonds, then your wealth at the end of year $n+1$ is $W_{n+1} = (ap + (1-p)V_n)W_n$. Suppose that $V, V_1, V_2, \ldots$ are i.i.d. with $EV_n < \infty$ and $EV_n^{-2} < \infty$.
(i) Show that $n^{-1} \log W_n \rightarrow_{a.s.} c(p)$.
(ii) Show that the limit $c(p)$ is a concave function of p. By computing $c'(0)$ and $c'(1)$, give conditions on V that guarantee that the optimal choice of p is in $(0, 1)$.
(iii) Suppose that $P(V = 1) = P(V = 4) = 1/2$. Find the optimal p as a function of a.

5. (Inversion of Laplace transforms.) Let P be a probability measure on the Borel subsets of $[0, \infty)$, and define its *Laplace transform* by

$\varphi(t) = \int_p^\infty e^{-tx} dP(x)$ for $t \in [0, \infty)$. Widder's inversion formula for P from φ is:

$$\lim_{n\to\infty} \sum_{k=0}^{[nz]} \frac{(-1)^k}{k!} n^k \varphi^{(k)}(n) = P([0, z)) \tag{1}$$

for $z \in [0, \infty)$ with $P(\{z\}) = 0$. Show that (1) holds via the following steps:

(a) Differentiation of the integral k times shows that

$$\varphi(t) = \int_0^\infty (-x)^k e^{-tx} dP(x) \, .$$

(b) Setting $t = n$, letting $z > 0$, multiplying across by $(-1)^k n^k / k!$, and summing on k yields

$$\sum_{k=0}^{[nz]} \frac{(-1)^k}{k!} n^k \varphi^{(k)}(n) = \int_0^\infty \sum_{k=0}^{[nz]} e^{-nx} \frac{(nx)^k}{k!} dP(x) \, . \tag{2}$$

where $e^{-nx}\frac{(nx)^k}{k!} = P(S_n = k)$ and $S_n = Y_1 + \ldots + Y_n$ where $Y_1, Y_2, \ldots$ are i.i.d. Poisson(x).

(c) Use the weak law of large numbers and (2) to show that (1) holds,