Statistics 522, Final Exam

Wellner; 3/5/99

Instructons:

A. This is a "take-home" exam. You may use your notes or other books. If you make use of a result in any book, you should give an exact citation of the source of the result.

B. You must do this exam completely on your own, with absolutely **no discussion** with other students.

C. The exam is **due** on **Monday, March 15, at 12 noon**, in the Statistics Office. Please give your exam solutions to Cheryl.

1. (36 points)
A. Give an example of a martingale with $X_n \to_p 0$, but $X_n \not\to_{a.s.} 0$. [Hint: Set $X_0 = 0$, and define $X_1, \ldots$ as follows: $P(X_k = \pm 1 | X_{k-1} = 0) = 1/(2k)$, $P(X_k = 0 | X_{k-1} = 0) = 1 - 1/k$; $P(X_k = kX_{k-1} | X_{k-1}) = (1/k) 1_{[X_{k-1} \neq 0]}$, $P(X_k = 0 | X_{k-1}) = (1 - 1/k) 1_{[X_{k-1} \neq 0]}$.]
B. Give an example of a martingale $\{X_n, \mathcal{A}_n\}$ with $X_n \to_{a.s.} -\infty$. [Hint: let $X_n = Y_1 + \cdots + Y_n$ where Y_i are independent with $E(Y_i) = 0$ and $P(Y_i = -1) = 1 - \epsilon_i$, then find an appropriate choice of the ϵ_i.]

2. (48 points).
A. Prove the following:
Lemma: If $\sigma-$fields $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, a $\sigma-$field, and $Y_n \to_1 Y$, then $E(Y_n|\mathcal{F}_n) \to_1 E(Y|\mathcal{F}_\infty)$.
B. Prove the following:
Lemma: Suppose that $\sigma-$fields $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, a $\sigma-$field, $Y_n \to_{a.s.} Y$, and $|Y_n| \leq Z$ where $Z \in L_1$. Show that $E(Y_n|\mathcal{F}_n) \to_{a.s.} E(Y|\mathcal{F}_\infty)$.
C. If $\{X_n\}$ is uniformly integrable and $X_n \to_{a.s.} X$, then $X_n \to_1 X$, and A above shows that $E(X_n|\mathcal{F}) \to_1 E(X|\mathcal{F})$. Find an example to show that $E(X_n|\mathcal{F})$ need not converge a.s.
[Hint: Let $Y_1, Y_2, \ldots$ and $Z_1, Z_2, \ldots$ be independent rv's with $P(Y_n = 1) = 1/n$, $P(Y_n = 0) = 1 - 1/n$, $P(Z_n = n) = 1/n$, $P(Z_n = 0) = 1 - 1/n$. Consider $X_n \equiv Y_n Z_n$ and $\mathcal{F} = \sigma[Y_1, Y_2, \ldots]$.]

3. (32 points).
Suppose that $Z_1, Z_2, \ldots$ are i.i.d. $N(0,1)$ rv's. Let $S_n \equiv \sum_{k=1}^n Z_k$, and define
$$Y_n \equiv \exp(aS_n - bn)\,.$$
A. Prove that $Y_n \to_{a.s.} 0$ if and only if $b > 0$.
B. For $r \geq 1$, prove that $Y_n \to_r 0$ if and only if $r < 2b/a^2$.

Do **either** problem 4 **or** problem 5.

4. (42 points).
Suppose that $X_1, X_2, \ldots$ are i.i.d non-negative random variables with $E(X_1) > 1$. Let $M_n \equiv \prod_{j=1}^n X_j$. Let $\mathcal{A}_n \equiv \sigma[M_1, \ldots, M_n]$.
A. Show that $\{M_n, \mathcal{A}_n\}$ is a sub-martingale.
B. Find the Doob decomposition of $\{M_n, \mathcal{A}_n\}$.
C. If $P(X_1 = 0) > 0$, show that $M_n \to_{a.s.} 0$ although $\{M_n\}$ is not integrable (i.e. L_1 bounded).

5. (42 points).
Let $X_1, X_2, \ldots$ be independent random variables with means $E(X_j) = 0$ and finite variances σ_j^2, $j = 1, 2, \ldots$. Let $S_n = X_1 + \ldots + X_n$, and $\mathcal{A}_n \equiv \sigma[X_1, \ldots, X_n]$, $n = 1, 2, \ldots$.
A. Show that $\{S_n^2, \mathcal{A}_n\}$ is a submartingale.
B. Evaluate the Doob decomposition of $\{S_n^2\}$.
C. If $\sum_{j=1}^\infty \sigma_j^2 < \infty$, show that the martingale $\{S_n\}$ converges a.s. (giving a martingale proof of part of the three-series theorem).

6. (40 points).
Let $X_1, \ldots, X_n$ be i.i.d. Poisson(1) random variables, set $S_n = X_1 + \ldots + X_n$, and $Z_n \equiv (S_n - n)/\sqrt{n}$. Prove Stirling's formula, $n! \sim \sqrt{2\pi n}(n/e)^n$, by showing that each of the following steps is valid:
A.
$$E\left(\frac{S_n - n}{\sqrt{n}}\right)^- = e^{-n}\sum_{k=0}^n \frac{n-k}{\sqrt{n}}\frac{n^k}{k!} = \frac{n^{n+1/2}e^{-n}}{n!}\,.$$
B. $Z_n \to_d Z \sim N(0,1)$.
C. $EZ_n^- \to EZ^- = 1/\sqrt{2\pi}$.
D. $n! \sim \sqrt{2\pi}n^{n+1/2}e^{-n} = \sqrt{2\pi n}(n/e)^n$.

7. (36 points).
Suppose that $X_1, X_2, \ldots$ are i.i.d. mean 0 rv's with $E(X_1^2) < \infty$. Let $S_n = \sum_{j=1}^n X_j$, $R_n = \sum_{j=1}^n S_j$, and $\mathcal{A}_n = \sigma[X_1, \ldots, X_n]$.
A. Show with $Y_n \equiv R_n - nS_n$, $\{Y_n, \mathcal{A}_n\}$ is a martingale.
B. Find the Doob-Meyer decomposition of the submartingale Y_n^2, and use it to compute $E(Y_n^2)$.