**Statistics 522, Midterm Exam**

Wellner; 2/17/99

1. (18 points). **Define** *three* of the following four terms:
(a) The conditional expectation of a random variable $X$ given a (sub-) sigma-field $\mathcal{D}$.
(b) A martingale, sub-martingale, and super-martingale.
(c) A stopping time (relative to a filtration $\mathcal{A}_n$).
(d) The compensator of a sub-martingale.

2. (27 points). Give careful **statements** of *three* of the following five theorems or results:
(a) The S-mg convergence theorem.
(b) The three-series theorem.
(c) The strong law of large numbers.
(d) The step-wise smoothing property of conditional expectations.
(e) The interpretation of conditional expectations in terms of an (orthogonal) projection onto $L_2(\Omega, \mathcal{G}, P)$ where $\mathcal{G} \subset \mathcal{A}$.

3. (42 points). Suppose that $X_0 = 1$, and let $X_n \sim \text{Uniform}(0, X_{n-1})$ for $n \geq 1$. Let $\mathcal{A}_n \equiv \sigma[X_0, X_1, \ldots, X_n]$ for $n = 0, 1, \ldots$.
(a) Show that with $Y_n \equiv 2^n X_n$, $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is a martingale, and hence that $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is a non-negative super-martingale.
(b) Apply the s-mg convergence theorem to the martingale $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$.

(c) There is no convergence theorem stated for a non-negative super-martingale in PfS, but based on what you know about the s-martingale convergence theorem and the reversed martingale convergence theorem, state a convergence theorem for non-negative supermartingales and apply it to $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$. What is the a.s. limit of $X_n$ in the present case?
(d) Is there any connection between the martingale $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ and Kakutani's product martingales?
(e) Use (d) to determine whether or not the martingale $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is uniformly integrable. Does convergence hold in $L_1$?

(f) Compute $E(X_{n+1}^2|\mathcal{A}_n)$ and $E(Y_{n+1}^2|\mathcal{A}_n)$.
(g) Use the computation in (f) to find a martingale related to $\{X_n^2\}$, and use it to compute $E(X_n^2)$ and $E(Y_n^2)$. Are either $\{X_n\}$ or $\{Y_n\}$ square-integrable?

**Do any two of the following three problems.**

4. (26 points). Suppose that $X \in L_2(P)$ and $\mathcal{D}$ is a sub-sigma field. The conditional variance of $X$ given $\mathcal{D}$ is defined by

$$Var(X|\mathcal{D}) = E\left\{(X - E(X|\mathcal{D}))^2|\mathcal{D}\right\} .$$

(a) Prove that

$$Var(X) = E[Var(X|\mathcal{D})] + Var(E(X|\mathcal{D})) .$$

(b) Interpret the formula in (a) geometrically.

5. (26 points). Suppose that $Z_1, Z_2, \ldots$ are i.i.d. $N(0,1)$ random variables. Let $\underline{Z} \equiv (Z_1, \ldots, Z_n)$, so that $\underline{Z} \sim N_n(0, I_n)$ where $I_n$ is the $n \times n$ identity matrix, and let $A$ be an orthogonal matrix (i.e. an $n \times n$ matrix with $A^T A = I_n$). Let $S^{n-1} \equiv \{\underline{x} \in R^n : |\underline{x}| = 1\}$.
(a) Show that $A\underline{Z} \sim N_n(0, I_n)$ for any orthogonal matrix $A$.
(b) Use the result of (a) to show that $\underline{Z}_n/|\underline{Z}_n| \sim$ Uniform$(S^{n-1})$. You may use the fact that the Uniform distribution on $S^{n-1}$ is the unique distribution which is invariant under orthogonal transformations.
(c) Show that $R_n = (Z_1^2 + \cdots + Z_n^2)^{1/2}$ satisfies $R_n/\sqrt{n} \rightarrow_{a.s.} 1$.
(d) Use (a) - (c) to show that if $\underline{Y}_n \sim$ Uniform$(S^{n-1})$, then for any fixed integer $k$ it follows that $\sqrt{n}(Y_{n1}, \ldots, Y_{nk}) \rightarrow_d N_k(0, I_k)$.

6. (26 points). Let $\xi_1, \xi_2, \ldots$ be i.i.d. Uniform$(0,1)$. Let $G(t) = t$ for $t \in [0,1]$, $G(t) = 0$ for $t \le 0$, $G(t) = 1$ for $t \ge 1$, be the Uniform$(0,1)$ distribution function. Let $\mathbb{G}_n(t) = n^{-1} \sum_{i=1}^n 1_{[0,t]}(\xi_i)$, the uniform empirical distribution function. Prove the Glivenko-Cantelli theorem for $\mathbb{G}_n$: $\|\mathbb{G}_n - G\|_\infty \equiv \sup_{0 \le t \le 1} |\mathbb{G}_n(t) - t| \rightarrow_{a.s.} 0$.