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On Some Results of M. I. Gordin: A Clarification of a Misunderstanding

Richard C. Bradley[1]

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M. I. Gordin proved a central limit theorem for some strictly stationary strongly mixing random sequences without the assumption of finite second moments. Because of a series of misunderstandings, his own correct formulation of the theorem has been essentially ignored, while an incorrect formulation has been discussed extensively and attributed to him in many references. This note explains in detail what has happened, in the hope of clearing up the misunderstandings.

KEY WORDS: Central limit theorem; strong mixing; infinite variance; Martingale theory.

1. INTRODUCTION

M. I. Gordin (Ref. 8, p. 174) proved a central limit theorem for some stationary random sequences satisfying the strong mixing condition. His own correct formulation of the theorem (Theorem 2.2 below) has been essentially ignored. As a result of a series of misunderstandings, an unintentional, incorrect formulation (Statement S below) has been discussed extensively and attributed to him in several references, in particular Refs. 1 and 10–12. This error was kindly pointed out to this writer by Professor Gordin himself and Professor I. A. Ibragimov (private communication).

Gordin[(8)] does not seem to be a well-known reference, even though the results therein are fundamental to the topic of strong mixing conditions and their connections with Martingale theory. Also, the difference between Gordin's result (the one that has been misstated) and the incorrect formulation is somewhat subtle. Therefore it seems worthwhile to explain in detail what has happened, in the hope that the misunderstandings will

[1] Department of Mathematics, Indiana University, Bloomington, Indiana 47405.

be cleared up. Section 2 will be devoted to a discussion of relevant results in Ref. 8, and then Section 3 will give a brief history of the misstatement.

2. SOME RESULTS IN GORDIN[8]

The first result is of a Martingale-theoretic nature. Suppose $(X_k, k \in \mathbb{Z})$ is a strictly stationary ergodic sequence of random variables on a probability space $(\Omega, \mathscr{F}, P)$, defined by $X_k(\omega) = X_0(T^k\omega)$, where T is an invertible measure-preserving ergodic transformation on $(\Omega, \mathscr{F}, P)$. Suppose $(\mathscr{M}_k, k \in \mathbb{Z})$ is a sequence of σ fields $\subset \mathscr{F}$ such that $\mathscr{M}_k \subset T^{-1}(\mathscr{M}_k) = \mathscr{M}_{k+1}$ for all $k \in \mathbb{Z}$. Define the partial sums $S_n := X_1 + \cdots + X_n$.

Theorem 2.1 (Ref. 8, p. 173). Under the above assumptions, suppose that $E\,|X_0| < \infty$,

$$\sum_{k=0}^{\infty} [E\,|E(X_0 \mid \mathscr{M}_{-k})| + E\,|X_0 - E(X_0 \mid \mathscr{M}_k)|] < \infty \tag{2.1}$$

and

$$\overline{\lim_{n \to \infty}}\; n^{-1/2} E\,|S_n| < \infty \tag{2.2}$$

Then $\lambda := \lim_{n \to \infty} n^{-1/2} E\,|S_n|$ exists, $0 \leqslant \lambda < \infty$; and as $n \to \infty$ the r.v. $n^{-1/2} S_n$ converges in distribution to the normal distribution with mean 0 and variance $(\pi/2)\,\lambda^2$ (degenerate if $\lambda = 0$).

Here and in Theorems 2.2 and 2.3 below, note that X_0 need not have finite second moment.

Gordin (Ref. 8, p. 173) mentions that one can prove Theorem 1 by applying a method in an earlier paper of his and using a result of Burkholder. The main steps in such a proof are as follows:

Step 1. Adapting the argument of Ref. 5, represent X_k by $X_k = Y_k + Z_k - Z_{k+1}$, where (Y_k) is a particular stationary ergodic sequence of Martingale differences and (Z_k) is a stationary sequence of integrable random variables.

Step 2. Using (2.2), ergodicity, and a result of Burkholder,[3] show that $EY_0^2 < \infty$.

Step 3. As in Ref. 5, apply to the sequence (Y_k) the Billingsley–Ibragimov central limit theorem for stationary ergodic sequences of square-integrable Martingale differences.

A detailed proof of Theorem 1 along these lines is given in Hall and Heyde (Ref. 9, Theorem 5.4); their proof of Step 2 is incorrect, but a detailed correct proof of Step 2 is given in Esseen and Janson (Ref. 4, Theorem 1).

Now we come to the results of Ref. 8 involving mixing conditions. As before, suppose $(X_k, k \in \mathbb{Z})$ is a strictly stationary sequence. For $-\infty \leqslant J \leqslant L \leqslant \infty$ let $\mathscr{F}_J^L$ denote the σ field generated by $(X_k, J \leqslant k \leqslant L)$. For each $n = 1, 2, 3, \ldots$ define the mixing coefficients

$$\alpha(n) := \sup |P(A \cap B) - P(A)\, P(B)|, \qquad A \in \mathscr{F}^0_{-\infty}, \qquad B \in \mathscr{F}_n^\infty$$

$$\phi(n) := \sup |P(B \mid A) - P(B)|, \qquad A \in \mathscr{F}^0_{-\infty}, \qquad B \in \mathscr{F}_n^\infty, \qquad P(A) > 0$$

The sequence (X_k) is "strongly mixing" if $\alpha(n) \to 0$ as $n \to \infty$, and "ϕ-mixing" (or "uniformly strong mixing") if $\phi(n) \to 0$ as $n \to \infty$.

Theorem 2.2 (Ref. 8, p. 174). Suppose (X_k) is strictly stationary, $EX_0 = 0$, and for some $p > 1$, $E\,|X_0|^p < \infty$ and $\sum_{n=1}^\infty \alpha(n)^{1-1/p} < \infty$. Suppose also that (2.2) holds. Then the conclusion of Theorem 2.1 holds.

Theorem 2.3 (Ref. 8, p. 174). Suppose (X_k) is strictly stationary, $E\,|X_0| < \infty$, $EX_0 = 0$, and $\sum_{n=1}^\infty \phi(n) < \infty$. Suppose also that (2.2) holds. Then the conclusion of Theorem 2.1 holds.

Gordin (Ref. 8, p. 174) mentions that Theorems 2.2 and 2.3 are proved by reducing them to Theorem 2.1. The argument is elementary, based on the simple inequalities

$$E\,|E(X_0 \mid \mathscr{F}_{-\infty}^{-n})| \leqslant 6 \cdot \|X_0\|_p \cdot \alpha(n)^{1-1/p} \qquad \text{for} \quad p > 1$$

and

$$E\,|E(X_0 \mid \mathscr{F}_{-\infty}^{-n})| \leqslant 2 \cdot \|X_0\|_1 \cdot \phi(n)$$

where $\phi(n)$, $n = 1, 2, \ldots$ refers here to the time-reversed process $(X_{-k}, k \in \mathbb{Z})$ instead of (X_k). For a proof of these two inequalities, see, e.g., Hall and Heyde [Ref. 9, p. 139, Eqs. (5.21)–(5.22), and p. 277, Corollary A.1].

3. THE MISSTATEMENT OF THEOREM 2.2

Theorem 2.2 (in its correct form) was proved by Gordin in the year 1969 and was given in Refs. 7 and 8). But by accident it was misstated by Gordin in Ref. 6). Then in Ibragimov and Linnik (Ref. 11, p. 420), in a description of relevant fresh discoveries of Gordin, because of a misunderstanding Theorem 2.2 was unintentionally misstated again, as follows:

Statement S (A Wrong Statement). Suppose (X_k) is strictly stationary and $EX_0 = 0$. Suppose that for some

$$\delta \geqslant 0, \qquad E|X_0|^{2+\delta} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha(n)^{(1+\delta)/(2+\delta)} < \infty$$

Suppose also that

$$(\operatorname{Var} S_n) \asymp n \qquad \text{as } n \to \infty \tag{3.1}$$

Then $S_n/\|S_n\|_2$ converges in distribution to $N(0, 1)$ as $n \to \infty$.

Hall and Heyde [Ref. 9, Corollary 5.3(ii)] also present Statement S, with (3.1) being replaced by the stronger assumption

$$\sigma^2 := \lim_{n \to \infty} n^{-1} ES_n^2 \quad \text{exists,} \qquad 0 < \sigma^2 < \infty \tag{3.1$'$}$$

But with either (3.1) or (3.1′), Statement S is incorrect. A counterexample [satisfying (3.1′)] was constructed by Herrndorf[10] for $\delta = 0$. Later, other similar counterexamples [satisfying (3.1′)] were constructed by Bradley (Ref. 1, Theorem 1), for arbitrary $0 < \delta < \infty$. In all of these counterexamples, the conclusion of Statement S fails to hold, but the conclusion of Theorem 2.2 holds with $\lambda = 0$. Thus, even if (3.1′) holds, the σ^2 in (3.1′) does not necessarily determine the value of λ in the conclusion of Theorem 2.2. (This was overlooked in the proof of Hall and Heyde [Ref. 9, Corollary 5.3(ii)).]

In several recent papers in which Statement S was extensively discussed—including Herrndorf,[10] Bradley (Ref. 1, p. 1315, lines 12–16; see also Ref. 2, p. 186, last four words, and p. 187, lines 1–2), and Peligrad (Ref. 12, p. 205, Comment 1.3)—Gordin's name was unfortunately attached to Statement S [with (3.1′)], with Ibragimov and Linnik (Ref. 11, p. 420) being cited as the reference.

Statement S has attracted the attention that should instead have been given to Theorem 2.2. In the literature on mixing conditions, the persistent attributing of Statement S [with (3.1) or (3.1′)] to Gordin has been an ongoing error and injustice. For having committed this error in Refs. 1 and 2, without having looked at Ref. 8, this writer wishes to apologize to Professor Gordin.

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