$$E\exp(i\kappa_l\lambda)=[E\exp(i\kappa_1\lambda)]^l$$
$$=\left[\sum_{m=1}^{\infty}e^{im\lambda}pq^{m-1}\right]^l$$
$$=\left[\frac{pe^{i\lambda}}{1-qe^{i\lambda}}\right]^l.$$

Substituting this into (3.14), we have that for each $\lambda\in[-\pi,\pi]$,

$$h(e^{i\lambda})=\operatorname{Re}p\cdot\left(a_0+2\sum_{l=1}^{\infty}a_l\cdot[pe^{i\lambda}/(1-qe^{i\lambda})]^l\right)$$
$$=\operatorname{Re}p\cdot g\left(\frac{pe^{i\lambda}}{1-qe^{i\lambda}}\right)=p\cdot u\left(\frac{pe^{i\lambda}}{1-qe^{i\lambda}}\right)$$

by (3.5) and (3.6). This and (3.13) give Lemma 3.2 for Case 1.

Case 2 The general case of continuous spectral density function $f(W,\cdot)$, as in the statement of Lemma 3.2.

Of course $f(W,\cdot)$ is also real, non-negative and symmetric on T.
Using Lemma 3.1, for each $n\geqq 1$ let f_n be a function of the form $f_n(z)=|p_n(z)|^2$ where $p_n(z)$ is a polynomial with real coefficients, such that

$$\forall\lambda\in[-\pi,\pi],\ |f_n(e^{i\lambda})-f(W,e^{i\lambda})|\leqq 1/n. \tag{3.15}$$

For each $n\geqq 1$ let $W^{(n)}:=(W_k^{(n)},\ k\in\mathbb{Z})$ be a strictly stationary real mean-zero finite-variance random sequence which is independent of the sequence V and has spectral density $f(W^{(n)},\cdot)=f_n(\cdot)$ on T. (One may have to enlarge the probability space, without changing the properties of the sequences V, W, κ and Y in Construction 2.3.) For each $n\geqq 1$ let $Y^{(n)}:=(Y_k^{(n)},\ k\in\mathbb{Z})$ be a (strictly stationary) sequence defined from $V,\kappa,W^{(n)}$ the same way that Y is defined from V, κ and W in Construction 2.3. [See Lemma 2.4(i).]

For each $k\in\mathbb{Z}$, by (3.15) and dominated convergence,

$$\lim_{n\to\infty}EW_0^{(n)}W_k^{(n)}=\lim_{n\to\infty}\int_{-\pi}^{\pi}e^{ik\lambda}f_n(e^{i\lambda})\,d\lambda$$

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$j\geqq 1$ [
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Hence

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$E\exp(i\kappa_1\lambda)]^l$

$$\sum_{m=1}^{\infty} e^{im\lambda}pq^{m-1}\Big]^l$$

$$\frac{pe^{i\lambda}}{1-qe^{i\lambda}}\Big]^l.$$

ave that for each $\lambda\in[-\pi,\pi]$,

$$a_l\cdot[pe^{i\lambda}/(1-qe^{i\lambda})]^l)$$

$$\frac{{}^{i\lambda}}{qe^{i\lambda}}\Big)=p\cdot u\left(\frac{pe^{i\lambda}}{1-qe^{i\lambda}}\right)$$

) give Lemma 3.2 for Case 1.

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$$^{i\lambda})-f(W,e^{i\lambda})|\leqq 1/n. \qquad (3.15)$$

$k\in\mathbb{Z}$) be a strictly stationary real sequence which is independent of lensity $f(W^{(n)},\cdot)=f_n(\cdot)$ on T. (One ility space, without changing the κ and Y in Construction 2.3.) For be a (strictly stationary) sequence ay that Y is defined from V, κ and ma 2.4(i).] ominated convergence,

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} e^{ik\lambda}f_n(e^{i\lambda})\,d\lambda$$

$$=\int_{-\pi}^{\pi} e^{ik\lambda}f(W,e^{i\lambda})\,d\lambda$$

$$=EW_0W_k.$$

Hence for each $j\geqq 1$, by Lemma 2.4(iii),

$$\lim_{n\to\infty} EY_0^{(n)}Y_j^{(n)}=\lim_{n\to\infty} p\cdot\sum_{l=1}^{j}(EW_0^{(n)}W_l^{(n)})\cdot P(\kappa_l=j)$$

$$=p\cdot\sum_{l=1}^{j}(EW_0W_l)\cdot P(\kappa_l=j)$$

$$=EY_0Y_j \qquad (3.16)$$

(3.17) This equality $\lim_{n\to\infty}EY_0^{(n)}Y_j^{(n)}=EY_0Y_j$ holds not only for $j\geqq 1$ [as in (3.16)], but also for $j=0$ by a similar argument [using Lemma 2.4(iii)] and for $j\leqq -1$ by "symmetry".

For each $n\geqq 1$ let u_n denote the (real) function on $\bar{D}$ which is harmonic on D, continuous on $\bar{D}$, and equal to f_n on T. By (3.15) and the maximum modulus principle, $\|u_n-u\|_\infty\leqq 1/n$ for each n. Also, each of the functions f_n satisfies the hypothesis of Case 1 [Eq. (3.3) and (3.4)] by a trivial argument. Hence for each $j\in\mathbb{Z}$, by (3.16)–(3.17), Case 1, and dominated convergence,

$$EY_0Y_j=\lim_{n\to\infty}EY_0^{(n)}Y_j^{(n)}=\lim_{n\to\infty}\int_{-\pi}^{\pi}e^{ij\lambda}p\cdot u_n\left(\frac{pe^{i\lambda}}{1-qe^{i\lambda}}\right)d\lambda$$

$$=\int_{-\pi}^{\pi}e^{ij\lambda}p\cdot u\left(\frac{pe^{i\lambda}}{1-qe^{i\lambda}}\right)d\lambda.$$

Hence the spectral density of Y is given by

$$f(Y,e^{i\lambda})=p\cdot u\left(\frac{pe^{i\lambda}}{1-qe^{i\lambda}}\right).$$

This completes the proof of Lemma 3.2.

In Section 4, in the proof of Theorem 1, when Construction 2.3 and Lemmas 2.4 and 3.2 are applied, the sequence W will be a stationary Gaussian sequence whose spectral density has a particular form ($1/2\pi$ times the function $H_c(\cdot)$ in Eq. (3.19) below). In preparation for this, a little more work is needed here.

Define the function G on T by

$$G(e^{i\lambda}) := \tfrac{1}{2} \sum_{k=2}^{\infty} \frac{1}{k \log k} (e^{ik\lambda} + e^{-ik\lambda})$$

$$= \sum_{k=2}^{\infty} \frac{1}{k \log k} (\cos k\lambda) \tag{3.18}$$

The sums here converge conditionally for $\lambda \in [-\pi, \pi] - \{0\}$, and diverge at $\lambda = 0$. In fact G is continuous on T except at $z = 1$ ($\lambda = 0$), and $\lim_{\lambda \to 0} G(e^{i\lambda}) = +\infty$. For this information, see [22, pp. 184 and 188, Theorems (1.8) and (2.15)].

For each $c > 0$, define the function H_c on T by

$$H_c(e^{i\lambda}) := \begin{cases} \exp[-c \cdot G(e^{i\lambda})] \text{ if } \lambda \in [-\pi, \pi] - \{0\} \\ 0 \text{ if } \lambda = 0. \end{cases} \tag{3.19}$$

By the above comments on G and elementary arguments, we have parts (i), (ii) and (iii) of the following lemma.

Lemma 3.3

i) $\forall c > 0$, *H_c is a real non-negative continuous symmetric function on T.*

ii) $\lim_{c \downarrow 0} [\sup_{-\pi \leq \lambda \leq \pi} H_c(e^{i\lambda})] = 1$.

iii) $\forall \lambda \in [-\pi, \pi] - \{0\}$, $\lim_{c \downarrow 0} H_c(e^{i\lambda}) = 1$.

iv) For every $\varepsilon > 0$ there exists a number $\gamma(\varepsilon) > 0$ such that the following statement holds: If $0 < c \leqq \gamma(\varepsilon)$, and $W := (W_k, k \in \mathbb{Z})$ is a stationary real Gaussian sequence with $EW_0 = 0$ and with spectral density $f(W, \cdot) = (1/2\pi) H_c(\cdot)$, then $\beta(W, 1) \leqq \varepsilon$, W is absolutely regular, $|EW_0^2 - 1| \leqq \varepsilon$, and $\forall k \neq 0$, $|EW_0 W_k| \leqq \varepsilon$.

Proof of Lemma 3.3(iv) Regardless of the value of c, one will automatically have absolute regularity by [16, p. 129, Theorem 8].

Theorem 1, when Construction 2.3 pplied, the sequence W will be a se spectral density has a particular $_c(\cdot)$ in Eq. (3.19) below). In prep- k is needed here.

$$\frac{1}{\log k}(e^{ik\lambda}+e^{-ik\lambda})$$

$$\frac{}{g\,k}(\cos k\lambda) \tag{3.18}$$

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$$e^{i\lambda})] \text{ if } \lambda\in[-\pi,\pi]-\{0\} \tag{3.19}$$

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a number $\gamma(\varepsilon)>0$ such that the $c\leqq\gamma(\varepsilon)$, and $W:=(W_k,\ k\in\mathbb{Z})$ is a with $EW_0=0$ and with spectral $\beta(W,1)\leqq\varepsilon$, W is absolutely regular, $\leqq\varepsilon$.

dless of the value of c, one will arity by [16, p. 129, Theorem 8].

The fact that $\beta(W,1)\leqq\varepsilon$ if c is sufficiently small, is a trivial corollary of [2, Lemmas 0.2 and 1.2] (see [2, p. 78] for the notations there). Also, for each $k\in\mathbb{Z}$,

$$\left|EW_0W_k-\int_{-\pi}^{\pi}e^{ik\lambda}\cdot\frac{1}{2\pi}d\lambda\right|$$

$$=\left|\int_{-\pi}^{\pi}e^{ik\lambda}\cdot\frac{1}{2\pi}H_c(e^{i\lambda})\,d\lambda-\int_{-\pi}^{\pi}e^{ik\lambda}\cdot\frac{1}{2\pi}d\lambda\right|$$

$$\leqq\int_{-\pi}^{\pi}\frac{1}{2\pi}|H_c(e^{i\lambda})-1|\,d\lambda.$$

The last integral here does not depend on k, and it converges to 0 as $c\downarrow 0$ by Lemma 3.3(ii) and (iii) and dominated convergence. The last part of Lemma 3.3(iv) follows. This completes the proof.

(3.20) For each $c>0$, let U_c be the (real) function on $\bar{D}$ which is harmonic on D, continuous on $\bar{D}$ and equal to H_c on T.

For each $c>0$, each $z\in D$,

$$|U_c(z)-1|=\left|\mathrm{Re}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\lambda}+z}{e^{i\lambda}-z}[H_c(e^{i\lambda})-1]\,d\lambda\right)\right|$$

$$\leqq\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|e^{i\lambda}+z|}{|e^{i\lambda}-z|}|H_c(e^{i\lambda})-1|\,d\lambda$$

$$\leqq\frac{1}{2\pi}\cdot\frac{2}{1-|z|}\int_{-\pi}^{\pi}|H_c(e^{i\lambda})-1|\,d\lambda \tag{3.21}$$

by [20, p. 255, Theorem 11.10] (and the trivial fact that $U_c(\cdot)-1$ is harmonic on D, continuous on $\bar{D}$ and equal to $H_c(\cdot)-1$ on T). By Lemma 3.3(ii) and (iii) and dominated convergence, $\lim_{c\downarrow 0}\int_{-\pi}^{\pi}|H_c(e^{i\lambda})-1|\,d\lambda=0$. Hence by (3.21), $\lim_{c\downarrow 0}U_c(z)=1$ for all $z\in D$, and this convergence is uniform on compact subsets of D.

(3.22) If $0<p<1$, $q=1-p$, and $0<\varepsilon<\pi$, then the set of points $pe^{i\lambda}/(1-qe^{i\lambda})$, $\lambda\in[-\pi,-\varepsilon]\cup[\varepsilon,\pi]$, is a closed, and hence compact, subset of D, and hence

$$\lim_{c\downarrow 0}\left[\sup_{\lambda\in[-\pi,-\varepsilon]\cup[\varepsilon,\pi]}\left|U_c\left(\frac{pe^{i\lambda}}{1-qe^{i\lambda}}\right)-1\right|\right]=0.$$

The following lemma lists some properties of the functions $U_c(\cdot)$ that will be used in Section 4.

LEMMA 3.4

i) Suppose $0<p<1$, $q=1-p$ *and* $0<\varepsilon<\pi$. *Then there exists a number* $\gamma^*(p,\varepsilon)>0$ *such that the following statement holds: If* $0<c\leqq\gamma^*(p,\varepsilon)$ *and the function u on T is defined by*

$$u(e^{i\lambda}) := U_c\left(\frac{pe^{i\lambda}}{1-qe^{i\lambda}}\right), \tag{3.23}$$

then u is real, non-negative, continuous and symmetric on T, and

$$u(e^{i\lambda})\begin{cases}\leqq 1+\varepsilon\ \forall\,\lambda\in[-\pi,\pi]\\ \geqq 1-\varepsilon\ \forall\,\lambda\in[-\pi,-\varepsilon]\cup[\varepsilon,\pi]\\ =0 \text{ if } \lambda=0.\end{cases}$$

ii) Suppose $0<p<1$, $q=1-p$, $\varepsilon>0$ *and g is a real continuous function on T. Then there exists a number* $\gamma^{**}(p,\varepsilon,g)>0$ *such that the following statement holds: If* $0<c\leqq\gamma^{**}(p,\varepsilon,g)$ *and the function u on T is defined by* (3.23), *then*

$$\left|\int_{-\pi}^{\pi} g(e^{i\lambda})\cdot u(e^{i\lambda})\,d\lambda-\int_{-\pi}^{\pi} g(e^{i\lambda})\,d\lambda\right|\leqq\varepsilon.$$

In Lemma 3.4(i), the symmetry of u on T follows from the symmetry of U_c on $\bar{D}$ (i.e., $U_c(\bar{z})=U_c(z)$), which in turn follows from the symmetry of H_c on T [see Lemma 3.3(i)]. The rest of Lemma 3.4(i) follows from Lemma 3.3(i), (ii) and (iii), the maximum modulus principle, (3.22), and basic properties of H_c and U_c stated above. Lemma 3.4(ii) follows from Lemma 3.4(i) and dominated convergence. (Of course, in Lemma 3.4(i), $u(\cdot)$ is real, non-negative, continuous and symmetric on T even if $c>\gamma^*(p,\varepsilon)$.)

$$ {}_{,\pi]}\left| U_c\left(\frac{pe^{i\lambda}}{1-qe^{i\lambda}}\right)-1\right|\Big]=0. $$

some properties of the functions $U_c(\cdot)$
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-p and $0<\varepsilon<\pi$. Then there exists a
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$$ =U_c\left(\frac{pe^{i\lambda}}{1-qe^{i\lambda}}\right), \tag{3.23} $$

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$\lambda\in[-\pi,\pi]$
$\lambda\in[-\pi,-\varepsilon]\cup[\varepsilon,\pi]$
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ts a number $\gamma^{**}(p,\varepsilon,g)>0$ such that the
$)<c\leqq\gamma^{**}(p,\varepsilon,g)$ and the function u on

$$ {}^{i\lambda})\,d\lambda-\int_{-\pi}^{\pi} g(e^{i\lambda})\,d\lambda\Big|\leqq\varepsilon. $$

metry of u on T follows from the
$(\bar{z})=U_c(z))$, which in turn follows from
ee Lemma 3.3(i)]. The rest of Lemma
(i), (ii) and (iii), the maximum modulus
roperties of H_c and U_c stated above.
Lemma 3.4(i) and dominated conver-
na 3.4(i), $u(\cdot)$ is real, non-negative,
. T even if $c\geqq\gamma^*(p,\varepsilon)$.)

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4. PROOF OF THEOREM 1

For each $L=1,2,3,\ldots$ let $\varepsilon_L>0$ be a number such that

$$ \begin{aligned} 1-2^{-L}&<1-2^{-(L+1/2)}-3\varepsilon_L\\ &<1-2^{-(L+1/2)}+3\varepsilon_L<1-2^{-(L+1)}. \end{aligned} \tag{4.1} $$

Obviously

$$ 0<\varepsilon_L<\frac{2^{-L}}{6}\text{ for all } L\geqq 1. \tag{4.2} $$

For each $L=1,2,3,\ldots$ we shall define a polynomial $p^{(L)}$ with real coefficients (its degree will be denoted J_L), a positive integer N_L, a positive number c_L, and a real non-negative continuous symmetric function u_L on T; such that for each $L\geqq 1$,

$$ \sum_{l=1}^{L}|p^{(l)}(e^{i\lambda})|^2u_l(e^{i\lambda})\begin{cases}\leqq 1-2^{-(L+1)}\ \forall\,\lambda\in[-\pi,\pi]\\ \geqq 1-2^{-L}\ \forall\,\lambda\in[-\pi,-\varepsilon_L]\cup[\varepsilon_L,\pi]\\ =0\text{ for }\lambda=0\end{cases} \tag{4.3} $$

The definition will be recursive.

To start off, let $p^{(1)}$ be the constant real polynomial defined by

$$ p^{(1)}(z):=(1-2^{-3/2})^{1/2}\text{ for all } z. \tag{4.4} $$

Then its degree is

$$ J_1=[\text{degree of } p^{(1)}]=0. \tag{4.5} $$

Define the positive integer N_1 by

$$ N_1:=2. \tag{4.6} $$

Keeping in mind that $0<\varepsilon_1<\pi$ by (4.2), let c_1 be a number such that

$$ 0<c_1\leqq\gamma(2^{-1}) \tag{4.7} $$

and

$$c_1 \leqq \gamma^*(2^{-1}, \varepsilon_1), \tag{4.8}$$

where the quantities $\gamma(\cdot)$ and $\gamma^*(\cdot,\cdot)$ are as in Lemmas 3.3(iv) and 3.4(i) respectively. Referring to (4.8) and Lemma 3.4(i), let u_1 be the real non-negative continuous symmetric function on T defined by

$$u_1(e^{i\lambda}) := U_{c(1)}\left(\frac{(1/2)e^{i\lambda}}{1-(1/2)e^{i\lambda}}\right). \tag{4.9}$$

Note that by (4.4) and (4.8)/(4.9)/Lemma 3.4(i) and (4.1), one has that

$$\begin{aligned}
&\forall \lambda \in [-\pi, \pi],\\
&\qquad |p^{(1)}(e^{i\lambda})|^2 u_1(e^{i\lambda}) \leqq (1-2^{-3/2})(1+\varepsilon_1)\\
&\qquad\qquad < 1-2^{-3/2}+\varepsilon_1 < 1-2^{-2};\\
&\forall \lambda \in [-\pi, -\varepsilon_1] \cup [\varepsilon_1, \pi],\\
&\qquad |p^{(1)}(e^{i\lambda})|^2 u_1(e^{i\lambda}) \geqq (1-2^{-3/2})(1-\varepsilon_1)\\
&\qquad\qquad > 1-2^{-3/2}-\varepsilon_1 > 1-2^{-1};
\end{aligned}$$

and $|p^{(1)}(e^{i\lambda})|^2 u_1(e^{i\lambda}) = 0$ for $\lambda = 0$. Thus the required Eq. (4.3) holds for $L=1$.

Now suppose that $L \geqq 2$, and that for each $l = 1, 2, \ldots, L-1$ the polynomial $p^{(l)}$ with real coefficients (and its degree J_l), the positive integer N_l, the positive number c_l, and the real non-negative continuous symmetric function u_l on T are already defined, such that

$$\sum_{l=1}^{L-1} |p^{(l)}(e^{i\lambda})|^2 u_l(e^{i\lambda}) \begin{cases} \leqq 1-2^{-L} \, \forall \lambda \in [-\pi, \pi] \\ \geqq 1-2^{-(L-1)} \, \forall \lambda \in [-\pi, -\varepsilon_{L-1}] \cup [\varepsilon_{L-1}, \pi] \\ = 0 \text{ for } \lambda = 0 \end{cases} \tag{4.10}$$

(i.e. the required Eq. (4.3) holds with L replaced by $L-1$).

Since the functions $|p^{(l)}(\cdot)|$ and $u_l(\cdot)$, $l = 1, \ldots, L-1$, are real,

$2^{-1}, \varepsilon_1),$ (4.8)

$\cdot,\cdot)$ are as in Lemmas 3.3(iv) and
8) and Lemma 3.4(i), let u_1 be the
metric function on T defined by

$$\left(\frac{(1/2)e^{i\lambda}}{1-(1/2)e^{i\lambda}}\right). \quad (4.9)$$

emma 3.4(i) and (4.1), one has that

$$\leqq (1-2^{-3/2})(1+\varepsilon_1)$$

$$\varepsilon_1 < 1-2^{-2};$$

$$[\varepsilon_1, \pi],$$

$$\geqq (1-2^{-3/2})(1-\varepsilon_1)$$

$$\varepsilon_1 > 1-2^{-1};$$

Thus the required Eq. (4.3) holds

that for each $l=1,2,\ldots,L-1$ the
its (and its degree J_l), the positive
r c_l, and the real non-negative
on T are already defined, such that

$$[-\pi,\pi]$$

$$\lambda\in[-\pi,-\varepsilon_{L-1}]\cup[\varepsilon_{L-1},\pi]$$

(4.10)

h L replaced by $L-1$).
d $u_l(\cdot)$, $l=1,\ldots,L-1$, are real,

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non-negative, continuous and symmetric on T, (4.10) implies that the function $1-2^{-(L+1/2)}-\sum_{l=1}^{L-1}|p^{(l)}(e^{i\lambda})|^2 u_l(e^{i\lambda})$ is also real, non-negative, continuous and symmetric on T. Using Lemma 3.1, let $p^{(L)}$ be a polynomial with real coefficients such that

$$\forall\lambda\in[-\pi,\pi], \left|1-2^{-(L+1/2)}-\sum_{l=1}^{L-1}|p^{(l)}(e^{i\lambda})|^2 u_l(e^{i\lambda}) - |p^{(L)}(e^{i\lambda})|^2\right| \leqq \varepsilon_L. \quad (4.11)$$

Multiplying $p^{(L)}$ by -1 if necessary, we assume that

$$p^{(L)}(1)\geqq 0. \quad (4.12)$$

Define its degree

$$J_L = \text{degree of } p^{(L)}. \quad (4.13)$$

From a well known property of the Fejer kernels [see (3.1)], we have

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(e^{i\lambda})|p^{(L)}(e^{i\lambda})|^2\,d\lambda = |p^{(L)}(e^{i\cdot 0})|^2 \geqq 1-2^{-(L+1/2)}-\varepsilon_L$$

where the last inequality holds by (4.10) and (4.11). Using this fact, let N_L be a positive integer such that

$$N_L \geqq 4^L, \quad (4.14)$$

$$N_L > N_{L-1}, \quad (4.15)$$

$$N_L^{1/2} \text{ is an integer}, \quad (4.16)$$

$$N_L^{1/2} > J_L, \quad (4.17)$$

and

$$\forall\, n \geqq N_L^{1/2},\ \frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(e^{i\lambda})|p^{(L)}(e^{i\lambda})|^2\, d\lambda$$

$$\geqq 1 - 2^{-(L+1/2)} - 2\varepsilon_L. \tag{4.18}$$

Keeping in mind that $0 < \varepsilon_L < \pi$ by (4.2) and $0 < N_L^{-1} < 1$ by (4.14), let c_L be a number such that

$$0 < c_L \leqq \gamma(2^{-L}/N_L^2), \tag{4.19}$$

$$c_L \leqq \gamma^*(N_L^{-1}, \varepsilon_L), \tag{4.20}$$

and

$$c_L \leqq \gamma^{**}\left(N_L^{-1}, \varepsilon_L, \frac{1}{2\pi}F_n(\cdot)|p^{(L)}(\cdot)|^2\right) \text{ for } n = N_L^{1/2}, N_L \tag{4.21}$$

where $F_n(\cdot)$ is as in (3.1) and the quantities $\gamma(\cdot)$, $\gamma^*(\cdot)$ and $\gamma^{**}(\cdot)$ are as in Lemmas 3.3(iv) and 3.4(i) and (ii). Referring to (4.20) and Lemma 3.4(i), let u_L be the real non-negative continuous symmetric function on T defined by

$$u_L(e^{i\lambda}) := U_{c(L)}\left(\frac{N_L^{-1}e^{i\lambda}}{1-(1-N_L^{-1})e^{i\lambda}}\right). \tag{4.22}$$

To complete this definition, all that remains is to check that the required Eq (4.3) holds for this value of L. First, by (4.11) and (4.1) we have that

$$\forall\, \lambda \in [-\pi, \pi],\ 1 - 2^{-(L+1/2)} - \varepsilon_L \leqq \sum_{l=1}^{L-1} |p^{(l)}(e^{i\lambda})|^2 u_l(e^{i\lambda})$$

$$+ |p^{(L)}(e^{i\lambda})|^2$$

$$\leqq 1 - 2^{-(L+1/2)} + \varepsilon_L$$

$$< 1. \tag{4.23}$$

$$\int_{-\pi}^{\pi} F_n(e^{i\lambda})|p^{(L)}(e^{i\lambda})|^2\, d\lambda$$

$${}^{1/2)} - 2\varepsilon_L. \qquad (4.18)$$

by (4.2) and $0<N_L^{-1}<1$ by (4.14), let

$$\leqq \gamma(2^{-L}/N_L^2), \qquad (4.19)$$

$${}^{*}(N_L^{-1}, \varepsilon_L), \qquad (4.20)$$

$${}_{n}(\cdot)|p^{(L)}(\cdot)|^2) \text{ for } n=N_L^{1/2}, N_L \qquad (4.21)$$

ıe quantities $\gamma(\cdot)$, $\gamma^*(\cdot)$ and $\gamma^{**}(\cdot)$ are (i) and (ii). Referring to (4.20) and l non-negative continuous symmetric

$$\left(\frac{N_L^{-1}e^{i\lambda}}{1-(1-N_L^{-1})e^{i\lambda}}\right). \qquad (4.22)$$

ıll that remains is to check that the value of L. First, by (4.11) and (4.1)

$${}^{1/2)} - \varepsilon_L \leqq \sum_{l=1}^{L-1} |p^{(l)}(e^{i\lambda})|^2 u_l(e^{i\lambda})$$

$$+|p^{(L)}(e^{i\lambda})|^2$$

$$\leqq 1-2^{-(L+1/2)}+\varepsilon_L$$

$$<1. \qquad (4.23)$$

Hence $|p^{(L)}(e^{i\lambda})|^2<1$ for all $\lambda\in[-\pi,\pi]$. Hence

$$|p^{(L)}(e^{i\lambda})|^2 u_L(e^{i\lambda}) - |p^{(L)}(e^{i\lambda})|^2 = |p^{(L)}(e^{i\lambda})|^2$$

$$\times[u_L(e^{i\lambda})-1]\begin{cases}\leqq \varepsilon_L \text{ for all } \lambda\in[-\pi,\pi],\\ \geqq -\varepsilon_L \text{ for all } \lambda\in[-\pi,-\varepsilon_L]\cup[\varepsilon_L,\pi]\end{cases}$$

by (4.20)/(4.22)/Lemma 3.4(i). Hence by (4.23),

$$\sum_{l=1}^{L} |p^{(l)}(e^{i\lambda})|^2 u_l(e^{i\lambda})$$

$$\begin{cases}\leqq 1-2^{-(L+1/2)}+2\varepsilon_L\ \forall\,\lambda\in[-\pi,\pi]\\ \geqq 1-2^{-(L+1/2)}-2\varepsilon_L\ \forall\,\lambda\in[-\pi,-\varepsilon_L]\cup[\varepsilon_L,\pi]\end{cases}$$

Using (4.1) we now get the two inequalities in (4.3) (for the specified values of λ). Also, $u_L(e^{i\cdot 0})=0$ (see e.g. (4.20)/(4.22)/Lemma 3.4(i) again), and hence by (4.10) we have the last part ("$=0$ for $\lambda=0$") in (4.3). Thus all of (4.3) holds, and this completes our recursive definition.

For each $L=1,2,3,\ldots$ let $V^{(L)}:=(V_k^{(L)},\ k\in\mathbb{Z})$ and $W^{(L)}:=(W_k^{(L)},\ k\in\mathbb{Z})$ be random sequences with the following properties:

(4.24) For each $L=1,2,3,\ldots$ $V^{(L)}$ is an i.i.d. sequence of $\{0,1\}$-valued r.v.'s such that $P(V_0^{(L)}=1)=N_L^{-1}$ and $P(V_0^{(L)}=0)=1-N_L^{-1}$. For each $L\geqq 1$, each $\omega\in\Omega$, $V_k^{(L)}(\omega)=1$ for infinitely many negative integers k and infinitely many positive integers k (a trivial technicality).

(4.25) For each $L=1,2,3,\ldots$ $W^{(L)}$ is a stationary real Gaussian sequence with $EW_0^{(L)}=0$ and with spectral density $f(W^{(L)},e^{i\lambda})=(1/2\pi)H_{c(L)}(e^{i\lambda})$ [see (3.19) and Lemma 3.3(i)].

(4.26) The sequences $V^{(1)}, W^{(1)}, V^{(2)}, W^{(2)},\ldots$ are independent of each other.

Note that $0<N_L^{-1}<1$ for each $L\geqq 1$, by (4.6) for $L=1$ and (4.14) for $L\geqq 2$. Hence for each $L\geqq 1$ the requirements in Construction 2.3 are met with p, q, V and W there being N_L^{-1}, $1-N_L^{-1}$, $V^{(L)}$ and $W^{(L)}$ here.

(4.27) For each $L=1,2,3,\ldots$ let $Y^{(L)}:=(Y_k^{(L)},\ k\in\mathbb{Z})$ be the random sequence defined from $V^{(L)}$ and $W^{(L)}$ the same way that the sequence Y was defined from V and W in Construction 2.3.

By Lemma 2.4(i), (ii) and (iii), one has that for each $L\geqq 1$, $Y^{(L)}$ is strictly stationary, and

$$\forall L\geqq 1,\ P(Y_0^{(L)}\neq 0)\leqq N_L^{-1} \tag{4.28}$$

and

$$\forall L\geqq 1,\ EY_0^{(L)}=0 \text{ and } E(Y_0^{(L)})^2<\infty. \tag{4.29}$$

(One can see that in (4.28) equality holds in our present context, but that is not important.) By (4.6)/(4.9)/(4.22), (3.20), (4.25) and Lemma 3.2, the spectral densities of the sequences $Y^{(L)}$ are given as follows:

$$\forall L\geqq 1, \forall \lambda\in[-\pi,\pi],\ f(Y^{(L)},e^{i\lambda})=\frac{1}{2\pi N_L}u_L(e^{i\lambda}). \tag{4.30}$$

For each $L=1,2,3,\ldots$, represent the polynomial $p^{(L)}$ by

$$p^{(L)}(z)=p_0^{(L)}+p_1^{(L)}z+p_2^{(L)}z^2+\cdots+p_{J(L)}^{(L)}z^{J(L)}. \tag{4.31}$$

(Recall (4.5)/(4.13) and the fact that the coefficients $p_j^{(L)}$ are real.) For each $L=1,2,3,\ldots$ define the (strictly stationary) sequence $X^{(L)}:=(X_k^{(L)},\ k\in\mathbb{Z})$ as follows:

$$\forall k\in\mathbb{Z},\ X_k^{(L)}:=N_L^{1/2}\sum_{j=0}^{J(L)}p_j^{(L)}Y_{k-j}^{(L)}. \tag{4.32}$$

Obviously by (4.29),

$$\forall L\geqq 1,\ EX_0^{(L)}=0 \text{ and } E(X_0^{(L)})^2<\infty. \tag{4.33}$$

By (4.30) and a well known formula for spectral densities of moving averages (see e.g. [9, p. 500, line -2]), the spectral densities of the sequences $X^{(L)}$ are given as follows:

$$\forall L\geqq 1,\ \forall \lambda\in[-\pi,\pi],\ f(X^{(L)},e^{i\lambda})=\frac{1}{2\pi}|p^{(L)}(e^{i\lambda})|^2u_L(e^{i\lambda}). \tag{4.34}$$

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$$P(Y_0^{(L)}\neq 0)\leqq N_L^{-1} \tag{4.28}$$

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By (4.26), (4.27) and (4.32), we have the following fact:

(4.35) The sequences $X^{(1)}, X^{(2)}, X^{(3)},\ldots$ are independent of each other.

The following comment is unnecessary but will render some elementary technicalities trivial. For each $L\geqq 2$,

$$|p^{(L)}(e^{i\lambda})|^2u_L(e^{i\lambda})\leqq 1-\sum_{l=1}^{L-1}|p^{(l)}(e^{i\lambda})|^2u_l(e^{i\lambda})$$

$$\begin{cases}\leqq 2^{-(L-1)} \text{ for } \lambda\in[-\pi,-2^{-L}]\cup[2^{-L},\pi]\\ \leqq 1 \text{ for } \lambda\in[-2^{-L},2^{-L}]\end{cases}$$

by (4.3) and (4.2), and hence

$$E(X_0^{(L)})^2=\int_{-\pi}^{\pi} f(X^{(L)},e^{i\lambda})\,d\lambda<2^{-(L-1)}+1\cdot 2\cdot 2^{-L}$$

$$=4\cdot 2^{-L}$$

by (4.33) and (4.34). Consequently

$$\sum_{L=1}^{\infty} E|X_0^{(L)}|\leqq\sum_{L=1}^{\infty}\|X_0^{(L)}\|_2<\infty. \tag{4.36}$$

Define the random sequence $X:=(X_k,\ k\in\mathbb{Z})$ as follows:

$$\forall k\in\mathbb{Z},\ X_k:=\sum_{L=1}^{\infty}X_k^{(L)}. \tag{4.37}$$

By (4.36) this sum is well defined (in fact absolutely convergent a.s.). Clearly X is strictly stationary. Our task now is to verify the properties (1.6)–(1.10) in Theorem 1 for this sequence X.

Proof of (1.6) $EX_0=0$ by (4.37), (4.36) and (4.33).

Next, for each $n\in\mathbb{Z}$,

$$\sum_{L=1}^{\infty}\sum_{M=1}^{\infty}E|X_0^{(L)}X_n^{(M)}|\leqq\sum_{L}\sum_{M}\|X_0^{(L)}\|_2\|X_0^{(M)}\|_2<\infty$$