STEIN'S METHOD VIA THE ZERO BIAS TRANSFORMATION

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1. The zero-bias transformation. Let X be a mean zero random variable with finite non-zero variance σ^2. Let X^* have the $X-$zero biased distribution: that is, X^* satisfies

$$E[Xf(X)] = \sigma^2 E[f'(X^*)]$$

for all differentiable f for which $E[Xf(X)]$ exists.

As Goldstein and Reinert (1997) show, the transformation $X \mapsto X^*$ (or, more precisely, the transformation $F \mapsto F^*$ where $X \sim F$, $X^* \sim F^*$) has several important properties. These are summarized in the following lemma.

Lemma. The transformation $X \mapsto X^*$ satisfies the following properties:
(i) $X \mapsto X^* = X$ if and only if $X \sim N(0, \sigma^2)$.
(ii) The distribution function F^* has density f^* which is unimodal about 0 and is given by

$$f^*(x) = \frac{1}{\sigma^2} \begin{cases} E[X1_{[X>x]}], & x \geq 0, \\ -E[X1_{[X\leq x]}], & x < 0 \end{cases}$$

(Note that since $E[X] = 0$, $E[X1_{[X>x]}] = -E[X1_{[X\leq x]}]$ for $x \in \mathbb{R}$.)
(iii) If X is symmetric about 0 (so $-X \stackrel{d}{=} X$), then X^* is symmetric about 0.
(iv) $\sigma^2 E[(X^*)^r] = E[X^{r+2}]/(r+1)$ for each integer $r \in \mathbb{N}$.
(v) Let $X_{n,1}, \ldots, X_{n,n}$ be independent mean zero random variables with

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$E[X_{n,i}^2] = \sigma_{n,i}^2$. Set $W_n = \sum_{i=1}^n X_{n,i}$ and $E[W_n^2] \equiv \sum_{i=1}^n \sigma_{n,i}^2 = 1$. Let I_n be a random integer with $P(I_n = i) = \sigma_{n,i}^2$. Then $W_n - X_{n,I_n} + X_{n,I_n}^* \stackrel{d}{=} W_n^*$.
(vi) Let X have mean zero with variance σ^2 and distribution function F. Let $(\hat{X}, \hat{X}')$ have joint distribution function $\hat{F}$ given by

$$d\hat{F}(\hat{x}, \hat{x}') = \frac{(\hat{x} - \hat{x}')^2}{2\sigma^2} dF(\hat{x}, \hat{x}').$$

Then, with $U \sim$ Uniform$(0, 1)$, $U\hat{X}, +(1 - U)\hat{X}' \stackrel{d}{=} X^*$.

As shown by Goldstein (2009), if $Y \sim$ Bernoulli(p), then $X \equiv Y - p$ (which has $E[X] = 0$ and $Var[X] = p(1-p)$ has $X^* \sim$ Uniform$(-p, 1-p) \stackrel{d}{=} U - p$ where $U \sim$ Uniform$(0, 1)$. Here we illustrate the transformation via several further examples.

Example 1. If $X \sim$ Logistic$(0, 1)$ (with $F(x) = 1/(1 + e^{-x})$, then the densities f and f^* are as follows:

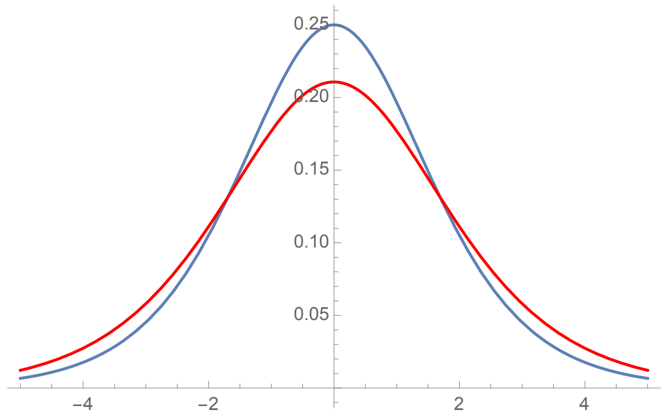

Fig 1. *The Logistic density f (blue) with the zero-bias transform density f^* (red)*

Example 2. If $X \sim$ centered Gumbel distribution (with $F(x) = 1 - \exp(-e^x)$), then the densities f and f^* are as follows:

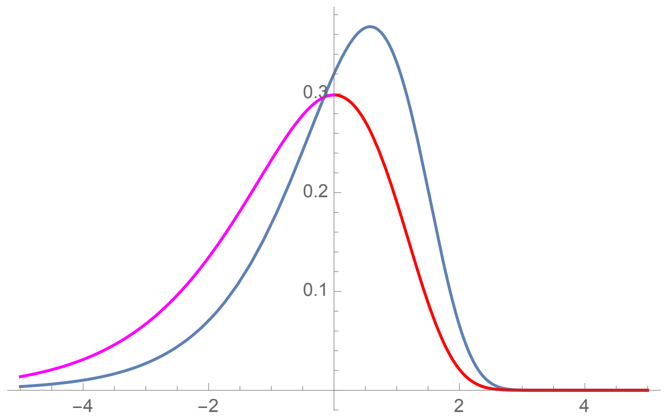

FIG 2. *The centered Gumbel density f (blue) with the zero-bias transform density f^* (red and magenta)*

Example 3. If $X \sim$ centered Weibull$(2, 1)$ distribution (with $F(x) = 1 - \exp(-x^2)$), then the densities f and f^* are as follows:

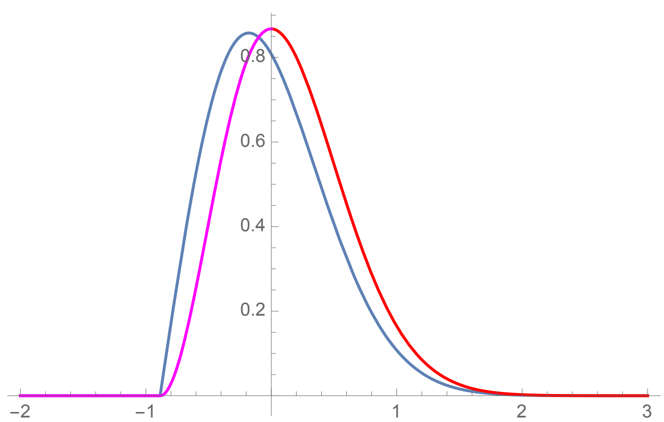

FIG 3. *The centered Weibull $(2, 1)$ density f (blue) with the zero-bias transform density f^* (red and magenta)*

Example 4. If $X \sim$ centered Gamma$(4,1)$ distribution (with $f(x) = x^3 \exp(-x)/\Gamma(4)$), then the densities f and f^* are as follows:

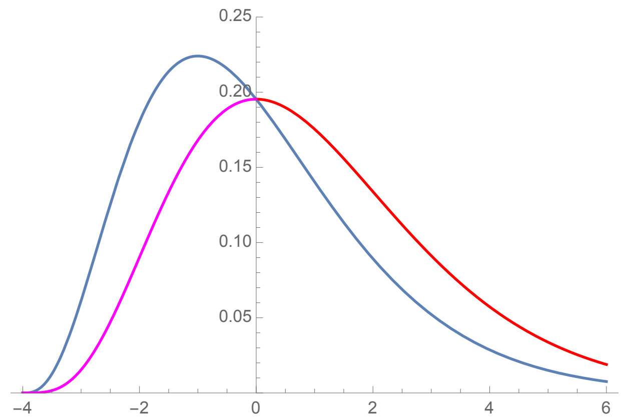

FIG 4. *The centered Gamma* $(4,1)$ *density* f *(blue) with the zero-bias transform density* f^* *(red and magenta)*

Example 5. If $X \sim$ centered Binomial$(16, 1/8)$, then the mass function and density function f and f^*, together with the $N(0, 16(1/8)(7/8)$ density, are as follows:

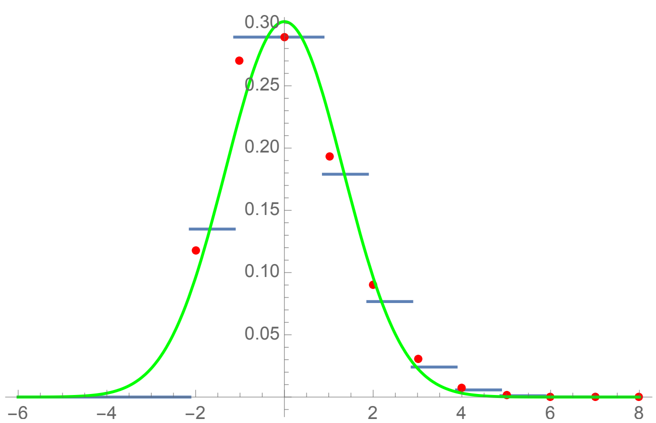

FIG 5. *The centered Binomial*$(16, 1/8)$ *mass function* f *(red) with the zero-bias transform density* f^* *(blue)*

Here are the distribution functions corresponding to the densities in Figure 5:

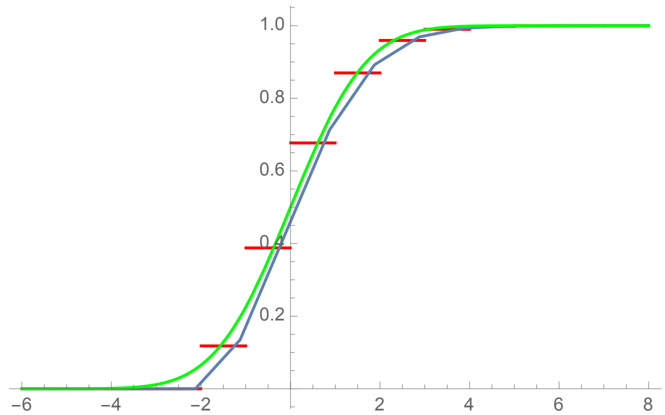

FIG 6. *The centered Binomial$(16, 1)$ distribution function f (red) with the zero-bias transform density f^* (blue), and the $N(0, 16(1/3)(2/3))$ distribution function (green)*

2. Solutions of the Stein equation: properties.

Let $Z \sim N(0, 1)$. Here we consider solutions to the Stein equation

$$f'(x) - xf(x) = h(x) - Eh(Z) \tag{1}$$

for several different choices of: $h(x) = 1_{(-\infty,t]}(x)$, $h(x) = 1_{(-\infty,t]}(x) + (1 - (x-t)/\epsilon)1_{[t,t+\epsilon]}(x)$, and for $h \in C_b^r(\mathbb{R})$ for $r \in \{0, 1, \ldots, \infty\}$. These are from Chen, Goldstein, and Shao (2011), pages 16-17 and 37 - 44.

Lemma 1. Let $h(x) = 1_{(-\infty,t]}(x)$ for $t \in \mathbb{R}$.
(i) Then the unique bounded solution of (1) is given by

$$f_t(x) = \frac{1}{\phi(x)} \left(\Phi(x \wedge t) - \Phi(t)\Phi(x) \right).$$

where $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ is the standard normal density function and $\Phi(x) = \int_{-\infty}^{x} \phi(y)dy$ is the standard normal density.
(ii) The solution $f_t(x)$ in (i) satisfies $x \mapsto xf_t(x)$ is an increasing function of x.
(iii) For all $x, y, t \in \mathbb{R}$,

$$\begin{aligned} |xf_t(x)| \le 1, &\qquad |xf_t(x) - yf_t(y)| \le 1, \\ |f_t'(s)| \le 1, &\qquad ||f_t'(x) - f_t'(y)| \le 1, \\ 0 < f_t(x) \le \min\{\sqrt{2\pi}/4,\ 1/|t|\}, & \end{aligned}$$

and

$$|(x+a)f_t(x+a) - (x+b)f_t(x+b)| \le \left(|x| + \sqrt{2\pi}/4\right)(|a| + |b|).$$

For the next two lemmas we note that the unique bounded solution of (1) is given by

$$
\begin{aligned}
f_h(x) &= \frac{1}{\phi(x)} \int_{-\infty}^{x} (h(y) - Eh(Z))\phi(y)dy \\
&= -\frac{1}{\phi(x)} \int_{x}^{\infty} (h(y) - Eh(Z))\phi(y)dy. \qquad (2)
\end{aligned}
$$

Lemma 2. Let $h(x) = 1_{(-\infty,t]}(x) + (1 - (x-t)/\epsilon)1_{[t,t+\epsilon]}(x)$ where $\epsilon > 0$. (This is the linearly smoothed version of the indicator function in Lemma 1). Then the unique bounded solution $f \equiv f_h$ given by (2) satisfies:
(i) $0 \le f(x) \le 1$, $|f'(x)| \le 1$, $|f'(x) - f'(y)| \le 1$.
(ii) $|f'(x+u) - f'(x)| \le |u| \left(1 + |x| + a^{-1} \int_0^1 1_{[y,y+a]}(x+ru)dr\right)$.

Lemma 3. For a given $h : \mathbb{R} \to \mathbb{R}$, let f_h be the solution (2). Then:
(i) If h is bounded, then

$$
\|f_h\|_\infty \le \sqrt{\pi/2}\|h - Eh(Z)\|_\infty \quad \text{and} \quad \|f_h'\|_\infty \le 2\|h - Eh(Z)\|_\infty.
$$

(ii) If h is absolutely continuous, then

$$
\|f_h\|_\infty \le 2\|h'\|_\infty, \quad \|f_h'\|_\infty \le \sqrt{2/\pi}\|h'\|_\infty, \quad \text{and} \quad \|f_h''\|_\infty \le 2\|h'\|_\infty.
$$

3. Appendix.

Theorem 2.1 If $Eh(Y_n) \to Eh(Y)$ for all $h \in C_{c,0}^\infty$, then $Y_n \to_d Y$ where $C_{c,0}^\infty$ denotes the set of all functions with compact support which integrate to 0 and which have derivatives of all order.

The proof of Theorem 2.1 is based on considering the following set of functions. First, let ψ be defined by $\psi(x) = 1$ for $x < 0$, $\psi(x) = 0$ for $x \ge 1$, and

$$
\psi(x) = \int_x^1 \exp\left(-\frac{1}{y(1-y)}\right) dy \Big/ \int_0^1 \exp\left(-\frac{1}{y(1-y)}\right) dy
$$

for $0 \le x \le 1$. This ψ takes values in $[0, 1]$ and is infinitely differentiable. For $a < b \in \mathbb{R}$ and $u > 0$, define

$$
\psi_{a,b,u}(x) = \psi(u(x-b)) - \psi(u(x-a)+1),
$$

for $x \in \mathbb{R}$. Note that $\psi_{a,b,u}$ takes values in $[0,1]$ for all x, and it equals 1 for $x \in [a,b]$. Moreover,

$$\int_{-\infty}^{\infty} \psi_{a,b,u}(x)dx = (b-a) + \frac{1}{u}.$$

Thus if $\epsilon \in (0,1]$ we can let

$$d = -\frac{1}{u} + \epsilon^{-1}\left(\frac{1}{u} + (b-a)\right),$$

and consider

$$\psi_{a,b,u,\epsilon}(x) = \psi_{a,b,u}(x) - \epsilon\psi_{b+2/u,b+2/u+d,u}(x) \tag{3}$$

These functions are in $C^{\infty}_{c,0}$. If we start with a linear smooth $\tilde{\psi}$ of the indicator $1_{(-\infty,0]}(x)$ instead of ψ, the resulting functions $\tilde{\psi}_{a,b,u,\epsilon}$ are depicted in Figure 7.

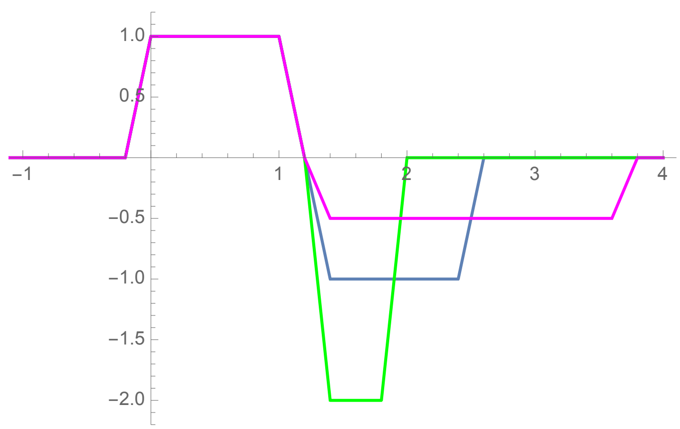

FIG 7. *The functions $\tilde{\psi}_{a,b,u,\epsilon}$ for $a=0$, $b=1, u=5, \epsilon \in \{1, 1/2, 2\}$*

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