7 Strong Markov Property for Sums of IID RVs

Let $X_1, X_2, \ldots$ be iid and let $S_n \equiv X_1 + \cdots + X_n$. Let $\mathrm{S} \equiv (S_1, S_2, ..)$.

Definition 7.1 The integer valued rv N is a *stopping time* for the sequence of rvs $S_1, S_2, \ldots$ if $[N = k] \in \mathcal{F}(S_1, \ \ldots, \ S_k)$ for all $k \geq 1$. It is elementary that

(1) $\quad \mathcal{F}_N \equiv \mathcal{F}(S_k : k \leq N)$

(2) $\quad \equiv \{A \in \mathcal{F}(\mathrm{S}) : A \cap [N = k] \in \mathcal{F}(S_1, \ldots, S_k) \text{ for all } k \geq 1\} = (\text{a } \sigma\text{-field}),$

since it is clearly closed under complements and countable intersections. (Clearly, $[N = k]$ can be replaced by $[N \leq k]$ in the definition of $\mathcal{F}_N$ in (2).)

Proposition 7.1 Both N and S_N are $\mathcal{F}_N$-measurable.

Proof. Now, to show that $[N \leq m] \in \mathcal{F}_N$ we consider $[N \leq m] \cap [N = k]$ equals $[N = k]$ or $\emptyset$, both of which are in $\mathcal{F}(\mathsf{S})$; this implies $[N \leq m] \in \mathcal{F}_N$. Likewise,

(a) $\quad [S_N \leq x] \cap [N = k] = [S_k \leq x] \cap [N = k] \in \mathcal{F}(S_1, \ \ldots, \ S_k),$

implying that $[S_N \leq x] \in \mathcal{F}_N$. □

Theorem 7.1 (The strong Markov property) If N is a stopping time, then the increments continuing from the random time

(3) $\quad \tilde{S}_k \equiv S_{N+k} - S_N, \ k \geq 1,$

have the same distribution on $(R_\infty, \ \mathcal{B}_\infty)$ as does $S_k, k \geq 1$. Moreover, defining $\tilde{S} \equiv (\tilde{S}_1, \tilde{S}_2, \ \ldots)$,

(4) $\quad \mathcal{F}(\tilde{S}) \equiv \mathcal{F}(\tilde{S}_1, \tilde{S}_2, \ \ldots)$ is independent of $\mathcal{F}_N$ (hence of N and S_N).

Proof. Let $B \in B_\infty$ and $A \in \mathcal{F}_N$. Now,

(a) $\quad P([\tilde{S} \in B] \cap A) = \sum_{n=1}^{\infty} P([\tilde{S} \in B] \cap A \cap [N = n])$

$\quad = \sum_{n=1}^{\infty} P([(S_{n+1} - S_n, \ S_{n+2} - S_n, \ \ldots) \in B] \cap (A \cap [N = n]))$

$\quad$ with $A \cap [N = n] \in \mathcal{F}(S_1, \ \ldots, \ S_n)$

$\quad = \sum_{n=1}^{\infty} P([(S_{n+1} - S_n, \ S_{n+2} - S_n, \ \ldots) \in B]) P(A \cap [N = n])$

$\quad = P(\mathsf{S} \in B) \sum_{n=1}^{\infty} P(A \cap [N = n])$

(b) $\quad = P(\mathsf{S} \in B) P(A).$

Set $A = \Omega$ in (b) to conclude that $\tilde{S} \cong \mathsf{S}$. Then use $P(\tilde{S} \in B) = P(\mathrm{S} \in B)$ to rewrite (b) as

(c) $\quad P([\tilde{S} \in B] \cap A) = P(\tilde{S} \in B) P(A),$

which is the statement of independence. □

Exercise 7.1 (Manipulating stopping times) Let N_1 and N_2 denote stopping times relative to an $\nearrow$ sequence of σ-fields $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots$. Show that $N_1 \wedge N_2, N_1 \vee N_2, N_1 + N_2$, and $N_o \equiv i$ are all stopping times.

Definition 7.2 Define *waiting times* for return to the origin by

$$
(5) \qquad \begin{array}{lll}
W_1 \equiv \min\{n : S_n = 0\} & \text{with } W_1 = +\infty & \text{if the set is empty,} \\
\vdots & \vdots & \\
W_k \equiv \min\{n > W_{k-1} : S_n = 0\} & \text{with } W_k = +\infty & \text{if the set is empty.}
\end{array}
$$

Then define $T_k \equiv W_k - W_{k-1}$, with $W_0 \equiv 0$, to be the *interarrival times* for return to the origin.

Proposition 7.2 If $P(S_n = 0 \text{ i.o.}) = 1$, then $T_1\, T_2, \ldots$ are well-defined rvs and are, in fact, iid.

Proof. Clearly, each W_k is always an extended-valued rv, and the requirement $P(S_n = 0 \text{ i.o.}) = 1$ guarantees that $W_k(\omega)$ is well-defined for all $k \geq 1$ for a.e. ω.

Now, $T_1 = W_1$ is clearly a stopping time. Thus, by the strong Markov property, T_1 is independent of the rv $\tilde{S}^{(1)} \equiv \tilde{S}$ with kth coordinate $\tilde{S}_k^{(1)} \equiv \tilde{S}_k \equiv S_{T_1+k} - S_{T_1}$ and $\tilde{S}^{(1)} \equiv \tilde{S} \cong \vec{S}$. Thus T_2 is independent of the rv $\tilde{S}^{(2)}$ with kth coordinate $\tilde{S}_k^{(2)} \equiv \tilde{S}_{T_2+k}^{(1)} - \tilde{S}_{T_2}^{(1)} = S_{T_1+T_2+k} - S_{T_1+T_2}$ and $\tilde{S}^{(2)} \cong \tilde{S}^{(1)} \cong \vec{S}$. Continue with $\tilde{S}^{(3)}$, etc. [Note the relationship to interarrival times of a Bernoulli process.] □

Exercise 7.2 (Wald's identity) (a) Suppose $X_1, X_2, \ldots$ are iid with mean μ, and N is a stopping time with finite mean. Show that $S_n \equiv X_1 + \cdots + X_n$ satisfies

$$
(6) \qquad \mathrm{E}S_N = \mu \mathrm{E}N.
$$

(b) Suppose each X_k equals 1 or -1 with probability p or $1-p$ for some $0 < p < 1$. Then define the rv $N \equiv \min\{n : S_n \text{ equals } -a \text{ or } b\}$, where a and b are strictly positive integers. Show that N is a stopping time that is a.s. finite. Then evaluate the mean $\mathrm{E}N$. [Hint. $[N \geq k] \in \mathcal{F}(S_1, \ldots, S_{k-1})$, and is thus independent of X_k, while $S_N = \sum_{k=1}^{\infty} X_k 1_{[N \geq k]}$.]