6 Embedding a RV in Brownian Motion

Let $a, b > 0$. For a Brownian motion $\mathbb{S}$ on $(C_\infty, \mathcal{C}_\infty)$, we define

$$(1)\qquad \tau \equiv \tau_{ab} \equiv \inf\{t : \mathbb{S}(t) \in (-a, b)^c\}$$

to be the first time $\mathbb{S}$ hits either $-a$ or b. Call τ a *hitting time*. [Show that τ is a stopping time.] Note figure 6.1.

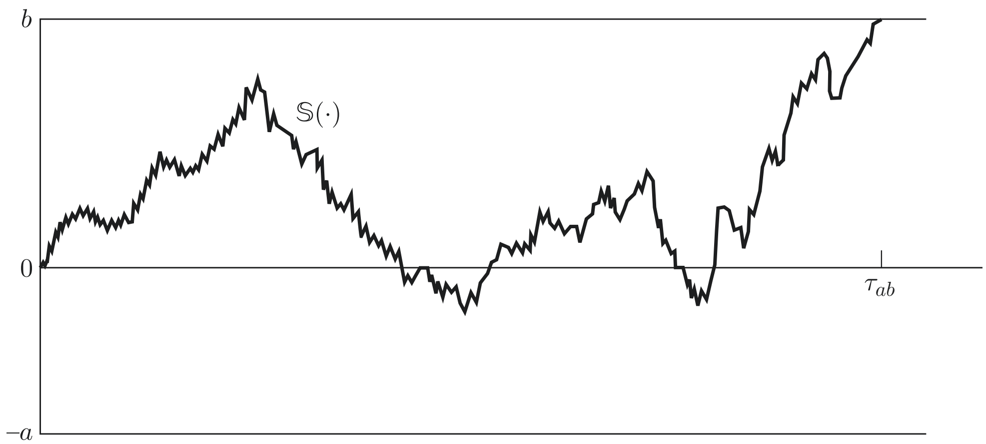

Figure 6.1 The stopping time τ_{ab}.

Theorem 6.1 (Embedding via τ_{ab}) Let $\tau \equiv \tau_{ab}$. Then:

$$(2)\qquad \mathrm{E}\mathbb{S}(\tau) = 0.$$

$$(3)\qquad P(\mathbb{S}(\tau) = -a) = b/(a+b) \qquad \text{and} \qquad P(\mathbb{S}(\tau) = b) = a/(a+b).$$

$$(4)\qquad \mathrm{E}\tau = ab = \mathrm{E}\mathrm{S}^2(\tau) \qquad \text{and} \qquad \mathrm{E}\tau^2 \le 4ab(a+b).$$

$$(5)\qquad \mathrm{E}\tau^r \le r\Gamma(r)2^{2r}\mathrm{E}\mathrm{S}^{2r}(\tau) \le r\Gamma(r)2^{2r}ab(a+b)^{2r-2} \qquad \text{for all } r \ge 1.$$

Definition 6.1 (Martingale) A process $\{M(t) : t \ge 0\}$ is a continuous parameter *martingale (mg)* if $\mathrm{E}|M(t)| < \infty$ for all t, M is adapted to the $\mathcal{A}_t$'s, and

$$(6)\qquad \mathrm{E}\{M(t)|\mathcal{A}_s\} =_{a.s.} M(s) \qquad \text{for all } 0 \le s \le t.$$

Definition 6.2 (Stopping time) If τ is a random time (just a rv that is ≥ 0) for which the event $[\tau \le t] \in \mathcal{A}_t$ for all t, then we call τ a *stopping time*.
Future theorem Let τ be a stopping time. With appropriate regularity conditions on a mg M, we can claim that

$$(7)\qquad \mathrm{E}M(\tau) = \mathrm{E}M(0).$$

Our present applications are simple special cases of a result called the *optional sampling theorem* for mgs. The general version will be proven in chapter 18. We will use it for such simple special cases now. □

Proof. The independent increments of $\mathbb{S}$ lead to satisfaction of the mg property stated in (6). Also, $\mathbb{S}$ is suitably integrable (we will see later) for (7) to hold (note (13.6.9) and (13.6.16)). Thus, with $p \equiv P(\mathbb{S}(\tau) = b)$, we have

(a) $$0 = \mathrm{E}\mathbb{S}(\tau) = bp - a(1-p), \qquad \text{or} \qquad p = a/(a+b).$$

Also, the process

(8) $$\{\mathbb{S}^2(t) - t : t \ge 0\} \qquad \text{is a mg adapted to the } \sigma\text{-fields } \mathcal{A}_t \equiv \sigma_t,$$

since

$$\begin{aligned}
\mathrm{E}\{\mathbb{S}^2(t) - t|\mathcal{A}_s\} &= \mathrm{E}\{[\mathbb{S}(t) - \mathbb{S}(s) + \mathbb{S}(s)]^2 - t|\mathcal{A}_s\} \\
&= \mathrm{E}\{[\mathbb{S}(t) - \mathbb{S}(s)]^2 + 2\mathbb{S}(s)[\mathbb{S}(t) - \mathbb{S}(s)] + \mathbb{S}^2(s) - t|\mathcal{A}_s\} \\
&= \mathrm{E}\{[\mathbb{S}(t) - \mathbb{S}(s)]^2\} + 2\mathbb{S}(s)\mathrm{E}\{\mathbb{S}(t) - \mathbb{S}(s)\} + \mathbb{S}^2(s) - t \\
&= t - s + 2\mathbb{S}(s) \cdot 0 + \mathbb{S}^2(s) - t
\end{aligned}$$

(b) $$= \mathbb{S}^2(s) - s.$$

This process is also suitably integrable, so that optional sampling can be used to imply $\mathrm{E}[\mathbb{S}(\tau)^2 - \tau] = 0$. Thus

(c) $$\mathrm{E}\tau = \mathrm{E}\mathbb{S}^2(\tau) = (-a)^2 \cdot b/(a+b) + b^2 \cdot a/(a+b) = ab.$$

We leave (5) to exercise 12.7.3 below. □

Theorem 6.2 (Skorokhod embedding of a zero-mean rv) Suppose X is a rv with df F having mean 0 and variance $0 \le \sigma^2 \le \infty$. Then there is a stopping time τ such that the stopped rv $\mathbb{S}(\tau)$ is distributed as X; that is,

(9) $$\mathbb{S}(\tau) \cong X.$$

Moreover,

(10) $$\mathrm{E}\tau = \mathrm{Var}[X] \qquad \text{and} \qquad \mathrm{E}\tau^2 \le 16\mathrm{E}X^4,$$

and for any $r \ge 1$ we have

(11) $$\mathrm{E}\tau^r \le K_r \mathrm{E}|X|^{2r} \quad \text{with} \qquad K_r \equiv r\Gamma(r)2^{4r-2}.$$

Proof. For degenerate F, just let $\tau \equiv 0$. Thus suppose F is nondegenerate. Let (A, B) be independent of $\mathbb{S}$, with joint df H having

(12) $$dH(a,b) = (a+b)dF(-a)dF(b)/EX^+ \qquad \text{for } a \ge 0, b > 0.$$

The procedure is to observe $(A, B) = (a, b)$ according to H, and then to observe τ_{ab}, calling the result τ. (Clearly, $\tau_{ab} = 0$ if $a = 0$ is chosen.) Note that $[\tau \le t]$ can be determined by (A, B) and $\{\mathbb{S}(s) : 0 \le s \le t\}$, and hence is an event in $\mathcal{A}_t \equiv \sigma[A, B, \mathbb{S}(s) : 0 \le s \le t\}$. For $t \ge 0$,

$$P(\mathbb{S}(\tau) > t) = \mathrm{E}(P\{\mathbb{S}(\tau) > t|A = a, B = b\})$$

(a) $$= \int_{[0,\infty)} \int_{(0,t]} 0 \cdot dH(a,b) + \int_{[0,\infty)} \int_{(t,\infty)} (a/(a+b))dH(a,b) \quad \text{by (3)}$$

(b) $\quad = \int_{(t,\infty)} \int_{[0,\infty)} \; a \; dF(-a) dF(b)/\mathrm{E}X^+ = \int_{(t,\infty)} dF(b) \mathrm{E}X^- / \mathrm{E}X^+$

(c) $\quad = 1 - F(t),$

since $EX = 0$ with X nondegenerate implies $\mathrm{E}X^+ = \mathrm{E}X^-$. Likewise, for $t \geq 0$,

(d) $\quad P(\mathbb{S}(\tau) \leq -t) = \int_{[0,t)} \int_{(0,\infty)} 0 \cdot dH(a,b) + \int_{[t,\infty)} \int_{(0,\infty)} (b/(a+b)) dH(a,b)$

(e) $\quad = \int_{[t,\infty)} \int_{(0,\infty)} b dF(b) dF(-a)/\mathrm{E}X^+ = \int_{[t,\infty)} dF(-a)$

(f) $\quad = F(-t).$

Thus $\mathbb{S}(\tau) \cong X$. Moreover,

$$\mathrm{E}\tau = \mathrm{E}(\mathrm{E}\{\tau | A = a, B = b\}) = \mathrm{E}(\mathrm{E}\{\mathbb{S}^2(\tau) | A = a, B = b\}) = \mathrm{E}\mathbb{S}^2(\tau)$$

(g) $\quad = \mathrm{E}X^2 = \mathrm{Var}[X].$

Note that $(a+b)^{2r-1} \leq 2^{2r-2}[a^{2r-1} + b^{2r-1}]$ by the C_r-inequality. Thus

$$\mathrm{E}\tau^r = \mathrm{E}(\mathrm{E}\{\tau^r | A = a, B = b\})$$

(h) $\quad \leq 2^{2r} r\Gamma(r) \mathrm{E}(AB(A+B)^{2r-2}) \quad \text{by (5)}$

$\quad \leq 2^{2r} r\Gamma(r) \mathrm{E}(AB(A+B)^{2r-1}/(A+B))$

(i) $$\leq r\Gamma(r) 2^{4r-2} \mathrm{E}\left(\frac{B}{A+B} A^{2r} + \frac{A}{A+B} B^{2r} \right)$$

(j) $\quad = K_r \mathrm{E}(\mathrm{E}\{\mathbb{S}^{2r}(\tau) | A = a, B = b\}) = K_r \mathrm{E}(\mathbb{S}^{2r}(\tau)) = K_r \mathrm{E}X^{2r},$

as claimed. □

7 Barrier Crossing Probabilities

For $-a < 0 < b$ we defined the *hitting time*

$$(1) \qquad \tau_{ab} \equiv \inf\{t \ : \ \mathbb{S}(t) \in (-a, b)^c\},$$

where $\mathbb{S}$ denotes Brownian motion on $(C_\infty, \mathcal{C}_\infty)$. We also considered the rv $\mathbb{S}(\tau_{ab})$, which is called Brownian motion *stopped* at τ_{ab}. We saw that it took on the two values b and $-a$ with the probabilities $p \equiv a/(a+b)$ and $q \equiv 1 - p = b/(a+b)$.

For $a > 0$ we define the stopping time (the *hitting time* of a)

$$(2) \qquad \tau_a \equiv \inf\{t : \mathbb{S}(t) \geq a\}.$$

[Now, $[\tau_a < c] = \cap_{q<a} \cup_{r<c} [\mathbb{S}(r) > q]$ (over rational p and q) shows that τ_a is a stopping time.] The LIL of (8.6.1) shows that both τ_{ab} and τ_a are finite a.s.

Theorem 7.1 (The reflection principle; Bachelier) Both

$$(3) \qquad P(\sup_{0\leq t\leq c} \mathbb{S}(t) > a) = P(\tau_a < c) = 2P(\mathbb{S}(c) > a)$$
$$= 2P(N(0,1) \geq a/\sqrt{c}) \qquad \text{for } a > 0 \qquad \text{and}$$

$$(4) \qquad P(\|\mathbb{S}\|_0^1 > a) = 4\sum_{k=1}^{\infty} P((4k-3)a < N(0,1) < (4k-1)a)$$

$$(5) \qquad = 1 - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2\pi^2}{8a^2}\right) \qquad \text{for } a > 0.$$

Proof. Define the stopping time $\tau_a' \equiv \tau_a \wedge c$, and note that $\tau_a = \tau_a'$ on the event $[\mathbb{S}(c) > a]$. Now, $[\tau_a' < c] \in \mathcal{A}_{\tau_a'}$ is independent of the Brownian motion $\{\mathbb{Y}(t) \equiv \mathbb{S}(\tau_a'+t) - \mathbb{S}(\tau_a') \ : \ t \geq 0\}$, by strong Markov, with $P(\tau_a' < \infty) = 1$. In (b) below we will use that $\mathbb{S}(\tau_a') = a$ on $[\mathbb{S}(c) > a]$. We have

$$P(\tau_a < c) = P(\tau_a' < c)$$

$$\text{(a)} \qquad = P([\tau_a' < c] \cap [\mathbb{S}(c) > a]) + P([\tau_a' < c] \cap [\mathbb{S}(c) < a]) + 0$$

$$\text{(b)} \qquad = P([\tau_a' < c] \cap [\mathbb{S}(c) - \mathbb{S}(\tau_a') > 0]) + P([\tau_a' < c] \cap [\mathbb{S}(c) - \mathbb{S}(\tau_a') < 0])$$

$$\text{(c)} \qquad = 2P([\tau_a' < c] \cap [\mathbb{S}(c) - \mathbb{S}(\tau_a) > 0]) \qquad \text{using the strong Markov property}$$

$$\text{(d)} \qquad = 2P(\mathbb{S}(c) > a),$$

since the events in (c) and (d) are identical.

The two-sided boundary of formula (4) follows from a more complicated reflec- tion principle. Let $A+ \equiv [\|\mathbb{S}^+\| > a] =$[$\mathbb{S}$ exceeds a somewhere on $[0,1]$] and $A_- \equiv [\|\mathbb{S}^-\| > a] =$ [$\mathbb{S}$ falls below $-a$ somewhere on $[0,1]$]. Though $[\|\mathbb{S}\| > a] = A_+ \cup A_-$, we have $P(\|\mathbb{S}\| > a) < P(A_+) + P(A_-)$, since we included paths that go above a and then below $-a$ (or vice versa) twice. By making the first reflection in figure 7.1, we see that the probability of the former event equals that of $A+- = [\|\mathbb{S}^+\| > 3a]$, while that of the latter equals that of $A_{-+} = [\|\mathbb{S}^-\| > 3a]$. But subtracting out these probabilities from $P(A^+) + P(A^-)$ subtracts out too much, since the path may then have recrossed the other boundary; we compensate for this by adding back in the probabilities of $A+-+ \equiv [\|\mathbb{S}^+\| > 5a]$ and $A_{-+-} \equiv [\|\mathbb{S}^-\| > 5a]$, which a second reflection shows to be equal to the appropriate probability. But we must continue this process ad infinitum. Thus

(e) $$P(\|\mathbb{S}\|_0^1 > a) = \begin{cases} P(A_+) - P(A_{+-}) + P(A_{+-+}) - \cdots + \\ P(A_-) - P(A_{-+}) + P(A_{-+-}) - \cdots \end{cases}$$

(f) $$= 2[P(A_+) - P(A_{+-}) + P(A_{+-+}) - \cdots] \qquad \text{by symmetry}$$
$$= 2\sum_{k=1}^{\infty}(-1)^{k+1} 2P(N(0,1) > (2k-1)a) \qquad \text{by (3)}$$

(g) $$= 4\sum_{k=1}^{\infty} P((4k-3)a < N(0,1) < (4k-1)a)$$

as claimed. The final expression (5) is left for the reader; it is reputed to converge more quickly. □

Exercise 7.1 Prove (5). (See Chung (1974, p. 223).)

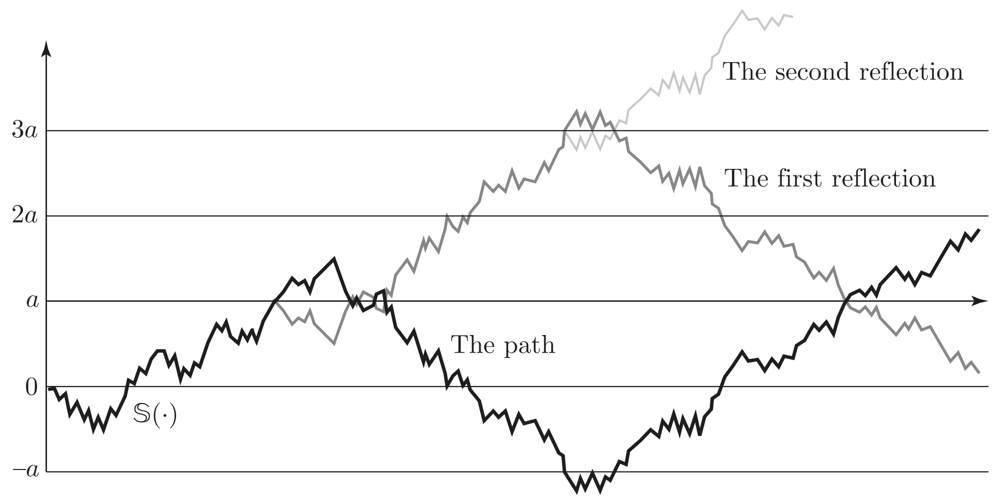

Figure 7.1 The reflection principle for Brownian motion.

Theorem 7.2 (The reflection principle for linear boundaries; Doob) Consider the line $ct+d$ with $c \ge 0, d > 0$. Then:

(6) $$P(\mathbb{S}(t) \ge ct + d \quad \text{for some } t \ge 0) = \exp(-2\,c\,d).$$

(7) $$P(|\mathbb{S}(t)| \ge ct + d \quad \text{for some } t \ge 0) = 2\sum_{k=1}^{\infty}(-1)^{k+1}\exp(-2k^2 cd).$$

Proof. Now, for any $\theta \ne 0$ the process

(8) $$\{V(t) \equiv \exp(\theta[\mathbb{S}(t) - \theta t/2]) \ : \ t \ge 0\} \qquad \text{is a mg} \quad (\text{with } V(0) \equiv 1).$$

This holds with $\sigma_t \equiv \sigma[\mathbb{S}(s) \ : \ s \le t]$ (using the mgf of a normal rv), since

$$\mathrm{E}\{V(t)|\sigma_s\} = \mathrm{E}\{\exp(\theta[\mathbb{S}(s) - \theta s/2] + \theta[\mathbb{S}(s,t] - \theta(t-s)/2])|\sigma_s\}$$

(a) $\qquad = V(s)\mathrm{E}\{\exp(\theta N(0,t-s))\}\ \exp(-\theta^2(t-s)/2)$

(b) $\qquad = V(s).$

Thus if we now redefine τ_{ab} as $\tau_{ab} \equiv \inf\{t\ :\ \mathbb{X}(t) \equiv \mathbb{S}(t) - \theta t/2 \in (-a,b)^c\}$, where we have $a > 0, b > 0$, then $V(t) = e^{\theta\mathbb{X}(t)}$. Hence the "future theorem" gives

(c) $\qquad 1 = \mathrm{E}V(\tau_{ab}) = P(\mathbb{X}(\tau_{ab}) = -a)e^{-\theta a} + P(\mathbb{X}(\tau_{ab}) = b)e^{\theta b},$

so that

(9) $\qquad P(\mathbb{X}(\tau_{ab}) = b) = (1 - e^{-\theta a})/(e^{\theta b} - e^{-\theta a})$

(d) $\qquad \to e^{-\theta b} \qquad \text{if } \theta > 0 \text{ and } \quad a \to \infty$

(e) $\qquad = e^{-2cd} \qquad \text{if } \theta = 2c \text{ and } b = d.$

But this same quantity also satisfies (by proposition 1.1.2)

(f) $\qquad P(\mathbb{X}(\tau_{ab}) = b) \to P(\mathbb{X}(t) \ge b \quad \text{for some } t) \qquad \text{as } a \to \infty$
$\qquad\qquad = P(\mathbb{S}(t) - \theta t/2 \ge b \quad \text{for some } t) = P(\mathbb{S}(t) \ge \theta t/2 + b \text{ for some } t)$

(g) $\qquad = P(\mathbb{S}(t) \ge ct + d \text{ for some } t) \qquad \text{if } c = \theta/2 \text{ and } d = b.$

Equating (g) to (e) (via (f) and (9)) gives (6). □

Exercise 7.2 Prove (7). (See Doob (1949).)

Theorem 7.3 (Kolmogorov–Smirnov distributions) Both

(10) $\qquad P(\|\mathbb{U}^{\pm}\| > b) = \exp(-2b^2) \qquad \text{for all } b > 0 \qquad \text{and}$

(11) $\qquad P(\|\mathbb{U}\| > b) = 2\sum_{k=1}^{\infty}(-1)^{k+1}\exp(-2k^2b^2) \qquad \text{for all } b > 0.$

Proof. Now, $\|\mathbb{U}^-\| \cong \|\mathbb{U}^+\|$ and

(12)
$$\begin{aligned} P(\|\mathbb{U}^+\| > b) &= P(\mathbb{U}(t) > b \quad \text{for some } 0 < t < 1) \\ &= P((1-t)\mathbb{S}(t/(1-t)) > b) \quad \text{for some } 0 \le t \le 1, \quad \text{by (12.3.16)} \\ &= P(\mathbb{S}(r) > b + rb \quad \text{for some } r \ge 0) \qquad \text{letting } r = t/(1-t) \end{aligned}$$

(a) $\qquad = \exp(-2b^2) \qquad \text{by theorem 7.2.}$

Likewise,

(b) $\qquad P(\|\mathbb{U}\| > b) = P(|\mathbb{S}(r)| > b + rb \text{ for some } r \ge 0)$

(c) $\qquad = 2\sum_{k=1}^{\infty}(-1)^{k+1}\exp(-2k^2b^2)$

by theorem 7.2. □

Exercise 7.3 (a) Prove (12.6.5) for $r = 2$.

(b) Prove (12.6.5) for integral r. (This is unimportant.)

[Hint. The $V_\theta \equiv \exp(\theta[\mathbb{S}(t) - \theta t^2/2]), t \geq 0$ of (8) are martingales on $[0, \infty)$. Differentiate formally under the integral sign in the martingale equality

(13) $\int_A V_\theta(t) dP = \int_A V_\theta(s) dP \qquad \text{for all } A \in \mathcal{A}_s.$

Then conclude that $[\partial^k / \partial\theta^k V_\theta(t)]|_{\theta=0}$ is a martingale for each $k \geq 1$. For $k = 4$ this leads to $\mathbb{S}^4(t) - 6t\mathbb{S}^2(t) + 3t^2 = t^2 H_4(\mathbb{S}(t)/\sqrt{t})$ being a martingale on $[0, \infty)$; here $H_4(t) = t^4 - 6t^2 + 3$ is the "fourth Hermite polynomial." The reader needs to work only with the single specific martingale in part (a); the rest of this hint is simply an intuitive explanation of how this martingale arises.]

8 Embedding the Partial Sum Process ^{o}

The Partial Sum Process

Let $X_{n1}, \ldots, X_{nn}$ be row-independent rvs having a common $F(0,1)$ distribution, and let $X_{n0} \equiv 0$. We define the *partial sum process* $\mathbb{S}_n$ on $(D, \mathcal{D})$ by

(1) $$\mathbb{S}_n(t) \equiv \frac{1}{\sqrt{n}} \sum_{i=0}^{[nt]} X_{ni} = \frac{1}{\sqrt{n}} \sum_{i=0}^{k} X_{ni} \qquad \text{for } \frac{k}{n} \le t < \frac{k+1}{n}, 0 \le k \le n$$

(or for all $k \ge 0$, in case the nth row is $X_{n1}, X_{n2}, \ldots$). Note that

(2) $$\begin{aligned} \mathrm{Cov}[\mathbb{S}_n(s), \mathbb{S}_n(t)] &= \sum\nolimits_{i=1}^{[ns]} \sum\nolimits_{j=1}^{[nt]} \mathrm{Cov}[X_{ni}, X_{nj}]/n \\ &= [n(s \wedge t)]/n \qquad \text{for } 0 \le s, t \le 1 \end{aligned}$$

for the greatest integer function $[\cdot]$. We suspect that $\mathbb{S}_n$ "converges" to $\mathbb{S}$. We will establish this shortly.

Embedding the Partial Sum Process

Notation 8.1 Let $\{\mathbb{S}(t) : t \ge 0\}$ denote a Brownian motion on $(C_\infty, \mathcal{C}_\infty)$. Then

(3) $$\mathbb{Z}_n(t) \equiv \sqrt{n}\,\mathbb{S}(t/n) \quad \text{for } t \ge 0 \qquad \text{is also such a Brownian motion.}$$

By using the Skorokhod embedding technique of the previous section repeatedly on the Brownian motion $\mathbb{Z}_n$, we may guarantee that for appropriate stopping times $\tau_{n1}, \ldots, \tau_{nn}$ (with all $\tau_{n0} \equiv 0$) we obtain that

(4) $$X_{nk} \equiv \mathbb{Z}_n(\tau_{n,k-1}, \tau_{nk}], \quad \text{for } 1 \le k \le n, \quad \text{are iid } F(0,1) \text{ rvs.}$$

Let $\mathbb{S}_n$ denote the partial sum process of these X_{nk}'s. Then, for $t \ge 0$ we have

(5) $$\begin{aligned} \mathbb{S}_n(t) &= \tfrac{1}{\sqrt{n}} \sum\nolimits_{k=1}^{[nt]} X_{nk} = \tfrac{1}{\sqrt{n}} \mathbb{Z}_n(\tau_{n,[nt]}) = \mathbb{S}\left(\tfrac{\tau_{n[nt]}}{n}\right) \\ &= \mathbb{S}\left(\frac{1}{n} \sum\nolimits_{k=1}^{[nt]} T_{nk}\right) = \mathbb{S}(I_n(t)) \end{aligned}$$

with $T_{nk} \equiv (\tau_{nk} - \tau_{n,k-1})$ and $I_n(t) \equiv \frac{1}{n}\tau_{n,[nt]}$. Observe that:

(6) $$X_{n1}, \ldots, X_{nn} \quad \text{are iid } F(0,1), \text{in each row.}$$

(7) $$T_{n1}, \ldots, T_{nn} \quad \text{are iid with means } = 1 = \mathrm{Var}[X], \text{in each row.}$$

(8) $$\mathrm{E}T_{nk}^r \le K_r \cdot \mathrm{E}|X_{nk}|^{2r}, \quad \text{with} \quad K_r \equiv r\Gamma(r)2^{4r-2}.$$ □

Theorem 8.1 (Skorokhod's embedding theorem) The partial sum process $\mathbb{S}_n$ on $(D, \mathcal{D})$ of row-independent $F(0,1)$ rvs formed as above satisfies

(9) $$\|\mathbb{S}_n - \mathbb{S}\| \to_p 0 \text{ as } n \to \infty.$$

Notice: The joint distributions of any $\mathbb{S}_m, \mathbb{S}_n$ in theorem 8.1 are not the same as they would be if formed from a single sequence of iid rvs. In fact, we have no idea of what these joint distributions may be. However, the partial sums of an iid sequence do not generally converge to their limit in the sense of $\to_p$, so we have gained a great deal via the embedding.

Theorem 8.2 (Embedding at a rate) Suppose that $\mathrm{E}X^4 < \infty$. Let I denote the identity function. Then for each $0 \le \nu < \frac{1}{4}$, the process $\mathbb{S}_n$ of (5) satisfies

(10) $\quad n^{\nu}\|(\mathbb{S}_n - \mathbb{S})/I^{1/2-\nu}\|_{1/n}^{1} = O_p(1).$

Proof. Consider theorem 8.1. Let I denote the identity function. Suppose we now show that

(a) $\quad \|I_n - I\|_0^1 = \sup_{0\le t\le 1} |\tau_{n,[nt]}/n - t| \to_p 0.$

Then on any subsequence n' where $\to_p 0$ in (a) may be replaced by $\to_{a.s.} 0$, the continuity of the paths of $\mathbb{S}$ will yield

(b) $\quad \|\mathbb{S}_{n'}(\cdot) - \mathbb{S}(\cdot)\| = \|\mathbb{S}(I_{n'}) - \mathbb{S}\| \to_{a.s.} 0,$

and thus (9) will follow. This is a useful argument; learn it. It therefore remains to prove (a). The WLLN gives

(c) $\quad I_n(t) = \tau_{n,[nt]}/n \to_p t \qquad$ for any fixed t.

Using the diagonalization technique, we can extract from any subsequence a further subsequence n' on which

(d) $\quad I_{n'}(t) \to_{a.s.} t \qquad$ for all rational t.

But since all functions involved are monotone, and since the limit function is continuous, this implies that a.s.

(e) $\quad I_{n'}(t) \to t \qquad$ uniformly on $F[0,1]$.

Thus (a) follows from (e), since every n has a further n' with the same limit. Thus the conclusion (9) holds.

In the proof just given, the conclusion (9) can trivially be replaced by

(f) $\quad \sup_{0\le t\le m} |\mathbb{S}_n(t) - \mathbb{S}(t)| \to_p 0.$

Appealing to exercise 12.1.6(b) for the definition of $\|\cdot\|_\infty$, we thus obtain

(11) $\quad \rho_\infty(\mathbb{S}_n, \mathbb{S}) \to_p 0 \quad$ on $\quad (C_\infty, \mathcal{C}_\infty),$

provided that the rvs $X_{n1}, X_{n2}, \ldots$ are appropriately iid $(0, \sigma^2)$. [We consider the proof of theorem 8.2 at the end of this section.] □

Let $g : (D, \mathcal{D}) \to (R, \mathcal{B})$ and let Δ_g denote the set of all $x \in D$ for which g is not $\|\cdot\|$-continuous at x. If there exists a set $\Delta \in \mathcal{D}$ having $\Delta_g \subset \Delta$ and $P(\mathbb{S} \in \Delta) = 0$, then we say that g is *a.s.*$\|\cdot\|$*-continuous* with respect to the process $\mathbb{S}$.

Theorem 8.3 (Donsker) Let $g : (D, \mathcal{D}) \to (R, \mathcal{B})$ denote an a.s. $\|\cdot\|$-continuous mapping that is $\mathcal{D}$-measurable. Then $g(\mathbb{S}_n) : (\Omega, \mathcal{A}, P) \to (R, \mathcal{B})$, and both

(12) $\quad g(\mathbb{S}_n) \to_p g(\mathbb{S}) \quad$ as $n \to \infty \qquad$ for the $\mathbb{S}_n$ of (5) and

(13) $\quad g(\mathbb{S}_n) \to_d g(\mathbb{S}) \quad$ as $\quad n \to \infty \qquad$ for any $\mathbb{S}_n$ having the same distribution.

($\mathcal{D}$-measurability is typically trivial, and hypothesizing it avoids the measurability difficulties discussed in section 12.1.) [Theorem 8.2 allows (13) for $\mathcal{D}$-measurable functionals g that are continuous in other $\|\cdot/q\|$-metrics.]

Proof. Now, $\|\mathbb{S}_n - \mathbb{S}\|$ is a $\mathcal{D}$-measurable rv, and $\|\mathbb{S}_n - \mathbb{S}\| \to_p 0$ for the $\mathbb{S}_n$ of (5). Thus any subsequence n' has a further subsequence n'' for which $\|\mathbb{S}_{n''} - \mathbb{S}\| \to 0$ for all $\omega \notin A''$, where $P(A'') = 0$. Moreover,

(a) $$P(A'' \cup [\mathbb{S} \in \Delta]) \le P(A'') + P(\mathbb{S} \in \Delta) = 0,$$

and if $\omega \notin A'' \cup [\mathbb{S} \in \Delta]$, then $g(\mathbb{S}_{n''}(\omega)) \to g(\mathbb{S}(\omega))$ holds, as $\|\mathbb{S}_{n''}(\omega) - \mathbb{S}(\omega)\| \to 0$ and g is $\|\ \|$-continuous at $\mathbb{S}(\omega)$. Thus $g(\mathbb{S}_n) \to_p g(\mathbb{S})$ as $n \to \infty$ for the $\mathbb{S}_n$ of (5). Thus $g(\mathbb{S}_n) \to_d g(\mathbb{S})$ for the $\mathbb{S}_n$ of (5), and hence of (13) also. [Note that we are dealing only with functionals for which the compositions $g(\mathcal{S}_n)$ and $g(\mathcal{S})$ are $(\Omega, \mathcal{A}, P) \to (R, \mathcal{B})$ measurable.] □

Example 8.1 Since the functionals $\|\cdot\|$ and $\|\cdot^+\|$ are a.s. $\|\cdot\|$-continuous,

(14) $$\|\mathbb{S}_n^+\| \to_d \|\mathbb{S}^+\| \quad \text{and} \quad \|\mathbb{S}_n\| \to_d \|\mathbb{S}\|.$$

The limiting distributions are those given in theorem 7.1. □

Exercise 8.1 Let $X_0 \equiv 0$ and $X_1, X_2, \ldots$ be iid $(0, \sigma^2)$. Define $S_k \equiv X_1 + \cdots + X_k$ for each integer $k \ge 0$.

(a) Find the asymptotic distribution of $(S_1 + \cdots + S_n)/c_n$ for an appropriate c_n.

(b) Determine a representation for the asymptotic distribution of the "absolute area" under the partial sum process, as given by $(|S_1| + \cdots + |S_n|)/c_n$.

The LIL

Recall the (8.6.1) LIL for a single sequence of iid $F(0,1)$ rvs $X_1, X_2, \ldots$ with partial sums $S_n \equiv X_1 + \cdots + X_n$; that is

(15) $$\overline{\lim}_{n\to\infty} |S_n|/\sqrt{2n \log\log n} = 1 \text{ a.s.}$$

The two LILs for Brownian motion (recall (12.3.7) and (12.3.18)) are

(16) $$\overline{\lim}_{t\to\infty} |\mathbb{S}(t)|/\sqrt{2t \log\log t} = 1 \text{ a.s. and}$$

(17) $$\overline{\lim}_{t\to 0} |\mathbb{S}(t)|/\sqrt{2t \log\log(1/t)} = 1 \text{ a.s.}$$

Notation 8.2 Define stopping times $T_1, T_2, \ldots$ (with $T_0 = 0$) having mean 1 for which the rvs

(18) $$X_k \equiv \mathbb{S}(\tau_{k-1}, \tau_k] \quad \text{are iid as } F.$$

Let $\tau_k \equiv T_0 + T_1 + \cdots + T_k$ for $k \ge 0$, and define the partial sums

(19) $$S_n \equiv \sum_{k=1}^n X_k = \mathbb{S}(\tau_n) = \mathbb{S}(n) + [\mathbb{S}(\tau_n) - \mathbb{S}(n)].$$

[Note that *this embedding differs* from that in notation 8.1. This one is based on a single sequence of rvs $X_1, X_2, \ldots$.] $\square$

Exercise 8.2 (The LIL) (a) First prove (15), while assuming that (16) is true. [Hint. By proposition 8.6.1, we want to show (roughly) that

$$(20) \qquad \begin{aligned} &|\mathbb{S}(\tau_n) - \mathbb{S}(n)|/\sqrt{2n \log\log n} \to_{a.s.} 0 \quad \text{or that} \\ &|\mathbb{S}(\tau_{[t]}) - \mathbb{S}(t)|/\sqrt{2t \log\log t} \to_{a.s.} \ 0. \end{aligned}$$

We will now make rigorous this approach to the problem. First apply the SLLN to $\tau[t]/t$ as $t \to \infty$. Then define $\Delta_k \equiv \sup\{|\mathbb{S}(t) - \mathbb{S}(t_k)| \ : \ t_k \le t \le t_{k+1}\}$, with $t_k \equiv (1+a)^k$ for some suitably tiny $a > 0$. Use a reflection principle and Mills' ratio to show that P ($\Delta_k \ge$ (an appropriate c_k)) $< \infty$. Complete the proof using Borel–Cantelli.]

(b) Now that you know how to deal with the "blocks" Δ_k, model a proof of (16) on the proof of proposition 8.6.1.

Proof for Embedding at a Rate*

Proof. Consider theorem 8.2. Let $d^2 \equiv \mathrm{Var}[T]$. Let $\mathrm{Log}k \equiv 1 \vee (\log k)$. Let $M \equiv M_\epsilon$ be specified below, and define

$$\text{(a)} \qquad A_n^c \equiv [\max_{1\le k\le n} |\sum_{i=1}^{k} (T_{ni} - 1)|/(d\sqrt{k}\mathrm{Log}k) \ge 2M/d].$$

Then the monotone inequality gives

$$A_n^c \subset [\max_{1\le k\le n} |\textstyle\sum_{i=1}^{k} \{(T_{ni}-1)/(d\sqrt{i}\mathrm{Log}i)\}| \ge M/d]$$

$$\text{(b)} \qquad \equiv [\max_{1\le k\le n} |\textstyle\sum_{i=1}^{k} Y_{ni}| \ge M/d],$$

where the Y_{ni}'s are independent with mean 0 and variance $(i\mathrm{Log}^2 i)^{-1}$. Thus the Kolmogorov inequality gives

$$\begin{aligned} &P(A_n^c) \le (d/M)^2 \mathrm{Var}[\sum_1^n Y_{ni}] = (d/M)^2 \sum_1^n (i\mathrm{Log}^2 i)^{-1} \\ (21) \qquad &\le (d/M)^2 \sum_1^\infty (i\mathrm{Log}^2 i)^{-1} \equiv (d/M)^2 v^2 < \epsilon^2 \quad \text{if} \quad M > dv/\epsilon \\ \text{(c)} \qquad &< \epsilon. \end{aligned}$$

Thus

$$\text{(d)} \qquad P(B_n) \equiv P\left(\max_{1\le k\le n} \frac{n^\nu |\mathbb{S}(\sum_1^{k_r} T_{ni}/n) - \mathbb{S}(k/n)|}{(k/n)^{1/2-\nu}} \ge \frac{2M}{\sqrt{dv}} \right)$$

$$\le P(B_n \cap A_n) + P(A_n^c)$$

(e) $$\le \sum_{k=1}^{n} P\left(\left[\frac{|\mathbb{S}(\Sigma_1^k T_{ni}/n) - \mathbb{S}(k/n)|}{[2M\sqrt{k}(\mathrm{Log}k)/n]^{1/2}} \ge \frac{2M}{\sqrt{dv}}\frac{k^{1/2-\nu}}{\sqrt{n}}\frac{1}{[2M\sqrt{k}(\mathrm{Log}k)/n]^{1/2}}\right] \cap A_n\right) + \epsilon$$

(f) $$\le \sum_{k=1}^{n} P(\sup_{0\le|r|\le a} |\mathbb{S}(r+k/n) - \mathbb{S}(k/n)|/\sqrt{a} \ge b) + \epsilon$$

with $a \equiv 2M\sqrt{k}(\mathrm{Log}k)/n$ (as in A_n in (a)),
and with $\ge b$ as on the right in (e)

(g) $$\le 3\sum_{k=1}^{n} P(\sup_{0\le r\le a} |\mathbb{S}(t, t+r]|/\sqrt{a} \ge b/3) + \epsilon \qquad \text{using (k) below}$$

(h) $$\le 12\sum_{k=1}^{n} P(N(0,1) \ge b/3) + \epsilon \qquad \text{by the reflection principle}$$

(i) $$\le 12\sum_{k=1}^{n} \exp(-(b/3)^2/2) + \epsilon \qquad \text{by Mills' ratio}$$

(22) $$\le 12\sum_{k=1}^{n} \exp\left(-\frac{M}{9dv}\frac{k^{1/2-2\nu}}{\mathrm{Log}k}\right) + \epsilon$$

(j) $$< 2\epsilon,$$

if $M \equiv M_\epsilon$ is large enough and if $0 \le \nu < \frac{1}{4}$ (this final step holds, since $\int_0^\infty \exp(-cx^\delta)\, dx \to 0$ as $c \to \infty$). The inequality (g) used

$$\sup_{0\le|r|\le a} |\mathbb{S}(r+k/n) - \mathbb{S}(k/n)|$$

(k) $$\le \sup_{0\le r\le a} |\mathbb{S}(r+k/n) - \mathbb{S}(k/n)| + 2\sup_{0\le r\le a} |\mathbb{S}(r+k/n-a) - \mathbb{S}(k/n-a)|$$

[with t in (g) equal to k/n or $k/n - a$, and with a as above (see (f))].

Now, $P(B_n) \le 2\epsilon$ shows that (10) is true, provided that the sup over all of $[1/n, 1]$ is replaced by the max over the points k/n with $1 \le k \le n$. We now "fill in the gaps". Thus (even a crude argument works here)

$$P(\sqrt{n}\max_{1\le k\le n-1} \sup_{0\le t\le 1/n} |\mathbb{S}(t+k/n) - \mathbb{S}(k/n)|/k^{1/2-\nu} \ge M)$$

$$\le \sum_{k=1}^{n-1} P\ (\mathbb{S}\|_0^{1/n} \ge Mk^{1/2-\nu}/\sqrt{n})$$

$$\le 4\sum_{k=1}^{n-1} P(N(0,1) \ge Mk^{1/2-\nu}) \qquad \text{by the reflection principle}$$

(23) $$\le 4\sum_{k=1}^{n-1} \exp(-M^2k^{1-2\nu}/2) \qquad \text{by Mills' ratio}$$

(l) $$< \epsilon,$$

if $M \equiv M_\epsilon$ is large enough and if $0 \le \nu < \frac{1}{2}$ (even). □

Exercise 8.3 Suppose $\mathrm{E}X^4 < \infty$. Show that the process $\mathbb{S}_n$ of (5) satisfies

(24) $$(n^{1/4}/\log n)\|\mathbb{S}_n - \mathbb{S}\| = O_p(1).$$

[Hint. Replace $n^\nu/(k/n)^{1/2-\nu}$ by $n^{1/4}/\log n$ in the definition of B_n in (d). Now determine the new form of the bounds in (20) and (21).] [While interesting and often quoted in the literature, this formulation has little value for us.]