Proposition 4.5 (Kallenberg) Let $\{\mathcal{A}_t : t \ge 0\}$ denote an $\nearrow$ sequence of σ-fields on the probability space $(\Omega, \mathcal{A}, P)$; that is, the $\mathcal{A}_t$'s form a filtration. Consider the null sets $\mathcal{N} \equiv \{N \in \mathcal{A} : P(N) = 0\}$. Define $\widehat{\mathcal{A}}_t \equiv \sigma[\mathcal{A}_t, \mathcal{N}]$.

(a) Then the $\nearrow$collection of σ-fields $\widehat{\mathcal{A}}_{t+}$ necessarily equals the completion $\widehat{\mathcal{A}_{t+}}$ of the right-continuous filtration $\mathcal{A}_{t+}$, and so forms an augmented filtration for $(\Omega, \widehat{\mathcal{A}}, P)$. Moreover, this is the minimal augmented filtration.

(b) If the $\mathcal{A}_t \equiv \sigma_t$ denote the histories of a right-continuous process $X : (\Omega, \mathcal{A}, P) \to (D_{[0,\infty)}, \mathcal{D}_{[0,\infty)})$, then the completion of the right-continuized histories necessarily forms the minimal augmented filtration. (That is, complete each $\sigma_{t+} \equiv \cap_{n=1}^{\infty} \sigma_{t+1/n}$.)

(c) If $S \le T$ a.s. then $\mathcal{A}_S \subset \mathcal{A}_T$ relative to the augmented filtration $\widehat{\sigma_{t+}} = \widehat{\sigma}_{t+}$.

Proof. It is trivial that $\widehat{\mathcal{A}_{t+}} \subset \widehat{\widehat{\mathcal{A}}_{t+}} = \widehat{\mathcal{A}}_{t+}$. To show the converse, consider a set $A \in \widehat{\mathcal{A}}_{t+}$. Then for each $n \ge 1$ we have $A \in \widehat{\mathcal{A}}_{t+1/n}$, so $P(A \Delta A_n) = 0$ for some set $A_n \in \mathcal{A}_{t+1/n}$. Note that $A^* \equiv \overline{\lim}_n A_n$ is in $\mathcal{A}_{t+}$, while $P(A \Delta A^*) = 0$ since $A \Delta A^* \subset \cup_1^\infty (A \Delta A_n) = \cup\{\text{nul1}\}_n = \{\text{null}\}$; thus $A \in \widehat{\mathcal{A}_{t+}}$. Thus the main claim in (a) is established. Let $\mathcal{F}_t$ denote any other augmented filtration for which all $\mathcal{F}_t \supset \mathcal{A}_t$. Then $\widehat{\mathcal{A}_{t+}} = \widehat{\mathcal{A}}_{t+} \subset \widehat{\mathcal{F}}_{t+} = \mathcal{F}_{t+} = \mathcal{F}_t$, as claimed. Part (b) follows at once. Part (c) follows from exercise 12.4.1 on properties of stopping times. □

Example 4.1 (Haeusler) Let both A and A^c be measurable subsets of some $(\Omega, \mathcal{A}, P)$ that have probability exceeding 0. Define $X_t(\omega)$ on $0 \le t \le 1$ to be identically 0 if $\omega \in A$ and to equal $(t - 1/2) \cdot 1_{[1/2,1]}(t)$ if $\omega \in A^c$. All paths of this X-process are continuous. Since X_t is always 0 for $0 \le t \le 1/2$, we have $\sigma_t = \{\emptyset, \Omega\}$ for $0 \le t \le 1/2$. However, $\sigma_t = \{\emptyset, A, A^c, \Omega\}$ for $1/2 < t \le 1$. These histories σ_t are not right continuous at $t = 1/2$. The right continuized histories σ_{t+} equal $\{\emptyset, \Omega\}$ for $0 \le t < 1/2$ and equal $\{\emptyset, A, A^c, \Omega\}$ for $1/2 \le t \le 1$. They are already complete, so $\widehat{\sigma}_{t+} = \widehat{\sigma_{t+}} = \sigma_{t+}$. Now, proposition 4.5(c) could be applied. □

5 Strong Markov Property

We now extend the strong Markov property (which was proved for discrete-time processes in section 8.6) to processes with stationary and independent increments.

Theorem 5.1 (Strong Markov property) Consider the stochastic process $X : (\Omega, \mathcal{A}, P) \to (D_{[0,\infty)}, \mathcal{D}_{[0,\infty)})$ adapted to right-continuous $\mathcal{A}_t$'s. Suppose that $X(0) = 0, X$ has stationary and independent increments, and suppose that the increment $X(t+s) - X(t)$ is independent of $\mathcal{A}_t$ for all $s \geq 0$. Let τ be an extended stopping time for the $\mathcal{A}_t$'s, and suppose $P(\tau < \infty) > 0$. For some $t \geq 0$ we define

(1) $$Y(t) \equiv \begin{cases} X(\tau + t) - X(\tau) & \text{on } [\tau < \infty], \\ 0 & \text{on } [\tau = \infty]. \end{cases}$$

Then $Y : ([\tau < \infty] \cap \Omega,\ [\tau < \infty] \cap \mathcal{A},\ P(\cdot|[\tau < \infty]))\ \to (D_{[0,\infty)}, \mathcal{D}_{[0,\infty)})$ and

(2) $$P(Y \in F|[\tau < \infty]) = P(X \in F) \quad \text{for all } F \in \mathcal{D}_{[0,\infty)}.$$

Moreover, for all $F \in \mathcal{D}_{[0,\infty)}$ and for all $A \in \mathcal{A}_\tau$, we have

(3) $$P([Y \in F] \cap A|[\tau < \infty]) = P([X \in F]) \times P(A|[\tau < \infty]).$$

Thus if $P(\tau < \infty) = 1$, then X and Y are equivalent processes and the process Y is independent of the σ-field $\mathcal{A}_\tau$.

Proof. That Y: $(\Omega \cap [\tau < \infty], \mathcal{A} \cap [\tau < \infty], P(\cdot|[\tau < \infty]) \to (D_{[0,\infty)}, \mathcal{D}_{[0,\infty)})$ follows from proposition 12.4.3. This proposition and exercise 12.4.1 show that

(4) $$\mathcal{A}'_t \equiv \mathcal{A}_{\tau+t} \text{ are } \nearrow \text{ and right continuous, with } Y \text{ adapted to the } \mathcal{A}'_t\text{'s.}$$

Case 1. Suppose the finite part of the range of τ is a countable subset $\{s_1, s_2, \ldots\}$ of $[0, \infty)$. Let $t_1, \ldots, t_m \geq 0$, let $B_1, \ldots, B_m$ be Borel subsets of the real line, and let $A \in \mathcal{A}_\tau$. Then

$$\begin{aligned} &P([Y(t_1) \in B_1, \ldots, Y(t_m) \in B_m] \cap A \cap [\tau < \infty]) \\ &\quad = \sum\nolimits_k P([Y(t_1) \in B_1, \ldots] \cap A \cap [\tau = s_k]) \\ &\quad = \sum\nolimits_k P([X(t_1 + s_k) - X(s_k) \in B_1, \ldots] \cap A \cap [\tau = s_k]) \\ &\quad = \sum\nolimits_k P(X(t_1 + s_k) - X(s_k) \in B_1, \ldots)P(A \cap [\tau = s_k]) \\ &\quad = P(X(t_1) \in B_1, \ldots) \sum\nolimits_k P(A \cap [\tau = s_k]) \end{aligned}$$

(a) $$\quad = P(X(t_1) \in B_1, \ldots, X(t_m) \in B_m)P(A \cap [\tau < \infty]),$$

where the third equality holds as $A \cap [\tau = s_k] = (A \cap [\tau \leq s_k]) \cap [\tau = s_k]$ is in $\mathcal{A}_{s_k}$, and is thus independent of the other event by the independent increments of X.
Putting $A = [\tau < \infty]$ in (a) yields

$$P(Y(t_1) \in B_1, \ldots, Y(t_m) \in B_m|[\tau < \infty])$$
(b) $$\quad = P(X(t_1) \in B_1, \ldots, X(t_m) \in B_m);$$

substituting (b) into (a) and dividing by $P(\tau < \infty)$ yields

$$P([Y(t_1) \in B_1, \ldots, Y(t_m) \in B_m] \cap A | [\tau < \infty])$$

$$\text{(c)} \qquad = P(Y(t_1) \in B_1, \ldots, Y(t_m) \in B_m | [\tau < \infty]) P(A | [\tau < \infty]).$$

Thus (b) and (c) hold for the class $\mathcal{G}$ of sets of the form $[Y(t_1) \in B_1, \ldots, Y(t_m) \in B_m]$ and for all sets A in $\mathcal{A}_\tau$. Since $\mathcal{G}$ generates $Y^{-1}(\mathcal{D}_{[0,\infty)})$, equation (b) implies (2). Since $\mathcal{G}$ is also closed under finite intersections (that is, it is a $\bar{\pi}-$system), (c) and proposition 7.1.1 imply the truth of (3).

Case 2. Now consider a general stopping time τ. For $n \geq 1$, define

$$\text{(d)} \qquad \tau_n \equiv \begin{cases} k/n & \text{for } (k-1)/n < \tau \leq k/n \text{ and } k \geq 1, \\ 1/n & \text{for } \tau = 0, \\ \infty & \text{for } \tau = \infty. \end{cases}$$

Note that $\tau_n(\omega) \searrow \tau(\omega)$ for $\omega \in [\tau < \infty]$. For $k/n \leq t < (k+1)/n$ we have

$$[\tau_n \leq t] = [\tau \leq k/n] \in \mathcal{A}_{k/n} \subset \mathcal{A}_t$$

(so that τ_n is a stopping time), and also for A in $\mathcal{A}_\tau$ that

$$A \cap [\tau_n \leq t] = A \cap [\tau \leq k/n] \in \mathcal{A}_{k/n} \subset \mathcal{A}_t$$

(so that $\mathcal{A}_\tau \subset \mathcal{A}_{\tau_n}$). Define

$$\text{(e)} \qquad Y_n(t) = X(\tau_n + t) - X(\tau_n) \quad \text{on } [\tau_n < \infty] = [\tau < \infty],$$

and let it equal 0 elsewhere. By case 1 results (b) and (c), both

$$\text{(f)} \qquad P(Y_n \in F | [\tau < \infty]) = P(X \in F) \quad \text{and}$$

$$\text{(g)} \qquad P([Y_n \in F] \cap A | [\tau < \infty]) = P(Y_n \in F | [\tau < \infty]) P(A | [\tau < \infty])$$

hold for all F in $\mathcal{D}_{[0,\infty)}$ and all A in $\mathcal{A}_\tau$ (recall that $\mathcal{A}_\tau \subset \mathcal{A}_{\tau_n}$ as shown above, and $[\tau < \infty] = [\tau_n < \infty]$). Let $(r_1, \ldots, r_m)$ denote any continuity point of the joint df of the finite dimensional random vector $(Y(t_1), \ldots, Y(t_m))$, and define

$$\text{(h)} \qquad \begin{aligned} G_n &\equiv [Y_n(t_1) < r_1, \ldots, Y_n(t_m) < r_m, \tau < \infty], \\ G &\equiv [Y(t_1) < r_1, \ldots, Y(t_m) < r_m, \tau < \infty], \\ G^* &\equiv [Y(t_1) \leq r_1, \ldots, Y(t_m) \leq r_m, \tau < \infty], \\ H &\equiv [X(t_1) < r_1, \ldots, X(t_m) < r_m]. \end{aligned}$$

By the right continuity of the sample paths, $Y_n(t) \to Y(t)$ for every t and every ω in $[\tau < \infty]$; thus

$$\text{(i)} \qquad G \subset \underline{\lim}\, G_n \subset \overline{\lim}\, G_n \subset G^*$$

Thus

$$\begin{aligned} P(G|\tau < \infty) &\leq P(\underline{\lim}\, G_n | [\tau < \infty]) \leq \underline{\lim}\, P(G_n | \tau < \infty) \quad \text{by (i), then DCT} \\ &= P(H) = \overline{\lim}\, P(G_n | \tau < \infty) \qquad \text{by using (f) twice} \\ &\leq P(\overline{\lim}\, G_n | \tau < \infty) \leq P(G^* | \tau < \infty) \qquad \text{by the DCT and (i)} \\ &\leq P(G|\tau < \infty) + \sum_{i=1}^m P(Y(t_i) = r_i | \tau < \infty) \end{aligned}$$

(j) $\qquad = P(G|\tau < \infty),$

since $(r_1, \ldots, r_m)$ is a continuity point. Thus (j) implies

(k) $\qquad P(G|\tau < \infty) = P(H),$

and this is sufficient to imply (2). Likewise, for $A \in \mathcal{A}_\tau \subset \mathcal{A}_{\tau_n}$,

$$\begin{aligned}
P(G \cap A|[\tau < \infty]) &\le P(\underline{\lim}\, G_n \cap A|\tau < \infty) \qquad \text{by (i)} \\
&\le \underline{\lim}\, P(G_n \cap A|\tau < \infty) \qquad \text{by the DCT} \\
&= \underline{\lim}\, P(G_n|\tau < \infty) P(A|\tau < \infty) \qquad \text{by (c), with } [\tau < \infty] = [\tau_n < \infty] \\
&= P(G|\tau < \infty) P(A|\tau < \infty) \qquad \text{by (j)} \\
&= \overline{\lim}\, P(G_n|\tau < \infty) P(A|\tau < \infty) \qquad \text{by (j)} \\
&= \overline{\lim}\, P(G_n \cap A|\tau < \infty) \qquad \text{by (c), with } [\tau < \infty] = [\tau_n < \infty] \\
&\le P(\overline{\lim}\, G_n \cap A|\tau < \infty) \le P(G^* \cap A|\tau < \infty) \qquad \text{by the DCT, then (i)} \\
&\le P(G \cap A|\tau < \infty) + \sum\nolimits_{i=1}^{m} P(Y(t_i) = r_i|\tau < \infty)
\end{aligned}$$

(l) $\qquad = P(G \cap A|\tau < \infty),$

since $(r_1, \ldots, r_m)$ is a continuity point. Thus (1) implies

(m) $\qquad P(G \cap A|\tau < \infty) = P(G|\tau < \infty) P(A|\tau < \infty);$

and using proposition 7.1.1 again, we see that this is sufficient to imply (3).

The final statement is immediate, since when $P(\tau < \infty) = 1$ we must have $P(A|\tau < \infty) = P(A)$ for all $A \in \mathcal{A}$. $\square$