

Poisson processes and a log-concave Bernstein theorem

by

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Abstract. We discuss interplays between log-concave functions and log-concave sequences. We prove a Bernstein-type theorem, which characterizes the Laplace transform of log-concave measures on the half-line in terms of log-concavity of the alternating Taylor coefficients. We establish concavity inequalities for sequences inspired by the Prékopa–Leindler and the Walkup theorems. One of our main tools is a stochastic variational formula for the Poisson average.

1. Introduction. Let $\varphi: [0, \infty) \rightarrow \mathbb{R}$ be a continuous function that is C^∞ -smooth on $(0, \infty)$. Its *alternating Taylor coefficients* are

$$(1) \quad a_t(n) = (-1)^n \frac{\varphi^{(n)}(t)}{n!} \quad (n \geq 0, t > 0).$$

A function whose alternating Taylor coefficients are non-negative is called a *completely monotone function*. Bernstein's theorem asserts that the alternating Taylor coefficients are non-negative if and only if there exists a finite, non-negative Borel measure μ on $[0, \infty)$ with

$$(2) \quad \varphi(t) = \int_0^\infty e^{-tx} d\mu(x) \quad (t \geq 0).$$

In other words, φ is the Laplace transform of the measure μ . See Widder [18] for proofs of Bernstein's theorem. We say that the alternating Taylor coefficients are *log-concave* if the sequence $(a_t(n))_{n \geq 0}$ is a log-concave sequence for any $t > 0$. This means that this sequence consists of non-negative numbers and for any $m, n \geq 0$ and $\lambda \in (0, 1)$ such that $\lambda n + (1 - \lambda)m$ is an integer,

$$(3) \quad a_t(\lambda n + (1 - \lambda)m) \geq a_t(n)^\lambda a_t(m)^{1-\lambda}.$$

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Equivalently, $a_t(n)^2 \geq a_t(n-1)a_t(n+1)$ for every $n \geq 1$, and the set of non-negative integers n for which $a_t(n) > 0$ is an interval of integers (either a finite interval of integers or an infinite one).

A measure μ on $[0, \infty)$ is *log-concave* if it is either a delta measure at a certain point, or else an absolutely continuous measure whose density $f : [0, \infty) \rightarrow \mathbb{R}$ is a log-concave function. Recall that a function $f : K \rightarrow \mathbb{R}$ for some convex set $K \subseteq \mathbb{R}^n$ is *log-concave* if f is non-negative and

$$f(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda} \quad \text{for all } x, y \in K, 0 < \lambda < 1.$$

THEOREM 1.1 (“Log-concave Bernstein theorem”). *Let $\varphi : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function that is C^∞ -smooth on $(0, \infty)$. Then the alternating Taylor coefficients of φ are log-concave if and only if φ takes the form (2) for a certain finite, log-concave measure μ .*

There are several known results about Laplace transforms of *log-convex* probability measures, rather than log-concave. In fact, Theorem 1.1 is the log-concave analog of Hirsch’s theorem, which analyzes the case where the alternating Taylor coefficients of φ are log-convex and non-increasing. Hirsch’s theorem states that this happens if and only if φ takes the form (2) for a measure μ whose density is non-increasing and log-convex, apart from an atom at the origin. See Hirsch [11] and Schilling, Song and Vondraček [15, Section 11.2] for a precise formulation and a proof of Hirsch’s theorem, and also Forst [10] and Sendov and Shan [16] for related results.

In Section 2 we prove Theorem 1.1 by using an inversion formula for the Laplace transform as well as the Berwald–Borell inequality [1, 2]. The latter inequality states that if one divides the Mellin transform of a log-concave measure on $[0, \infty)$ by the Gamma function, then a log-concave function is obtained. It directly implies the “if” part of Theorem 1.1. In the proof of Hirsch’s theorem from [15], the rôle of the Berwald–Borell inequality is replaced by the simpler Cauchy–Schwarz inequality. Theorem 1.1 admits the following corollary:

COROLLARY 1.2. *Let μ be a finite, non-negative Borel measure on $[0, \infty)$ and let φ be given by (2). Then μ is log-concave if and only if the function $|\varphi^{(n-1)}(t)|^{-1/n}$ is convex in $t \in (0, \infty)$ for every $n \geq 1$.*

In fact, in Theorem 1.1 it suffices to verify that the sequence $(a_t(n))_{n \geq 0}$ is log-concave for a sufficiently large t , as follows from the following:

PROPOSITION 1.3. *Let $\varphi : (0, \infty) \rightarrow \mathbb{R}$ be real-analytic, and define $a_t(n)$ via (1). Assume that $0 < r < s$ and that the sequence $(a_s(n))_{n \geq 0}$ is log-concave. Then the sequence $(a_r(n))_{n \geq 0}$ is also log-concave.*

Proposition 1.3 is proven in Section 3, alongside concavity inequalities related to log-concave sequences in the spirit of the Walkup theorem [17].

While searching for a Prékopa–Leindler type inequality for sequences, we found the following:

THEOREM 1.4. *Let $f, g, h, k : \mathbb{Z} \rightarrow \mathbb{R}$ satisfy*

$$f(x) + g(y) \leq h\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right) + k\left(\left\lceil \frac{x+y}{2} \right\rceil\right), \quad \forall x, y \in \mathbb{Z},$$

where $\lfloor x \rfloor$ is the lower integer part of $x \in \mathbb{R}$ and $\lceil x \rceil$ is the upper integer part. Then

$$(4) \quad \left(\sum_{x \in \mathbb{Z}} e^{f(x)}\right) \left(\sum_{x \in \mathbb{Z}} e^{g(x)}\right) \leq \left(\sum_{x \in \mathbb{Z}} e^{h(x)}\right) \left(\sum_{x \in \mathbb{Z}} e^{k(x)}\right).$$

Our proof of Theorem 1.4 is presented in Section 5 and it involves probabilistic techniques. It would be interesting to find a direct proof. However, we believe that the probabilistic method is not without importance in itself, and perhaps it is mathematically deeper than other components of this paper. The argument is based on a stochastic variational formula for the expectation of a given function with respect to the Poisson distribution. It is analogous to Borell’s formula from [4] which is concerned with the Gaussian distribution. The stochastic variational formula is discussed in Section 4.

Theorem 1.4 is reminiscent of the variant of the Prékopa–Leindler inequality obtained by Cordero and Maurey [8, Proposition 1.2], all the more so as their main tool is the stochastic formula of Borell mentioned above. Let us also mention that it is easy to tensorize Theorem 1.4. However, the inequality that we obtain is not entirely satisfactory as it depends on a rather arbitrary way of rounding coordinates. For instance, in dimension 2, one gets two formally different inequalities: The conclusion

$$\left(\sum_{x \in \mathbb{Z}^2} e^{f(x)}\right) \left(\sum_{x \in \mathbb{Z}^2} e^{g(x)}\right) \leq \left(\sum_{x \in \mathbb{Z}^2} e^{h(x)}\right) \left(\sum_{x \in \mathbb{Z}^2} e^{k(x)}\right)$$

holds if either

$$f(x) + g(y) \leq h\left(\left\lfloor \frac{x_1 + y_1}{2} \right\rfloor, \left\lfloor \frac{x_2 + y_2}{2} \right\rfloor\right) + k\left(\left\lceil \frac{x_1 + y_1}{2} \right\rceil, \left\lceil \frac{x_2 + y_2}{2} \right\rceil\right)$$

for every $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in \mathbb{Z}^2 , or if

$$f(x) + g(y) \leq h\left(\left\lfloor \frac{x_1 + y_1}{2} \right\rfloor, \left\lceil \frac{x_2 + y_2}{2} \right\rceil\right) + k\left(\left\lceil \frac{x_1 + y_1}{2} \right\rceil, \left\lfloor \frac{x_2 + y_2}{2} \right\rfloor\right)$$

for every $x, y \in \mathbb{Z}^2$.

The Berwald–Borell inequality (or Theorem 1.1) implies that when μ is a finite, log-concave measure on $[0, \infty)$ and $k \leq \ell \leq m \leq n$ are non-negative integers with $k + n = \ell + m$,

$$(5) \quad a_t(\ell)a_t(m) - a_t(k)a_t(n) \geq 0,$$

where $a_t(n) = \int_0^\infty (x^n/n!)e^{-tx} d\mu(x)$ is defined via (1) and (2). The following theorem shows that the left-hand side of (5) is not only non-negative, but it is in fact a completely monotone function of t :

THEOREM 1.5. *Let μ be a finite, log-concave measure on $[0, \infty)$. Then for any non-negative integers $k \leq \ell \leq m \leq n$ with $k + n = \ell + m$ there exists a finite, non-negative measure $\nu = \nu_{k,\ell,m,n}$ on $[0, \infty)$ such that for any $t > 0$,*

$$(6) \quad \int_0^\infty e^{-tx} d\nu(x) = a_t(\ell)a_t(m) - a_t(k)a_t(n),$$

where as usual $a_t(n) = \int_0^\infty (x^n/n!)e^{-tx} d\mu(x)$ is defined via (1) and (2).

Theorem 1.5 is proven in Section 3. Let us apply this theorem in a few examples. In the case where μ is an exponential measure, whose density is $t \mapsto \alpha e^{-\alpha t}$ on $[0, \infty)$, the measures ν from Theorem 1.5 vanish completely. In the case where μ is proportional to a Gamma distribution, the measures ν are also proportional to Gamma distributions. When μ is the uniform measure on the interval $[1, 2]$, the density of the measure $\nu = \nu_{0,1,1,2}$ from Theorem 1.5 is depicted in Figure 1. This log-concave density equals the convex function $(t-1)(t-2)/2$ in the interval $[2, 3]$, and it equals $(t-2)(4-t)$ in $[3, 4]$.

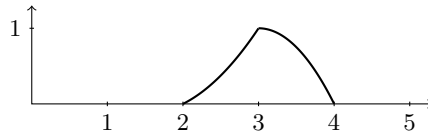


Fig. 1. The density of $\nu_{0,1,1,2}$ where μ is uniform on the interval $[1, 2]$

We suggest that the measure ν from Corollary 1.5 be referred to as the Berwald–Borell transform of μ with parameters (k, ℓ, m, n) . All Berwald–Borell transforms of log-concave measures that we have encountered so far were log-concave themselves. It is an interesting problem to characterize the family of measures ν which could arise as the Berwald–Borell transform of a log-concave measure on $[0, \infty)$. Such a characterization could lead to new constraints on the moments of log-concave measures on $[0, \infty)$ beyond the constraints posed by the Berwald–Borell inequality.

2. Proof of the log-concave Bernstein theorem. The proof of Theorem 1.1 combines ideas of Berwald from the 1940s with the earlier Post inversion formula for the Laplace transform. The “if” direction of Theorem 1.1 follows from:

LEMMA 2.1. *Let μ be a finite, log-concave measure on $[0, \infty)$. Assume that φ is given by (2). Then the alternating Taylor coefficients of φ are log-concave.*

Proof. In the case where $\mu = c\delta_0$ for some $c \geq 0$, we have $\varphi \equiv c$ and the alternating Taylor coefficients of φ are trivially log-concave. In the case where $\mu = c\delta_{x_0}$ for $x_0 > 0$ we have $\varphi(t) = ce^{-tx_0}$ and hence

$$a_t(n) = \frac{ce^{-tx_0}x_0^n}{n!}.$$

Since $a_t(n) > 0$ for every n and $a_t(n+1)/a_t(n) = x_0/(n+1)$ is non-increasing, this is indeed a log-concave sequence. In the case where μ has a log-concave density f , we denote $f_t(x) = e^{-tx}f(x)$ and observe that

$$(7) \quad \varphi^{(k)}(t) = \int_0^\infty (-x)^k e^{-tx} f(x) dx = \int_0^\infty (-x)^k f_t(x) dx \quad (k \geq 0, t > 0).$$

The function f_t is log-concave, and hence we may apply the Berwald–Borell inequality [1, 2]; see also [6, Theorem 2.2.5] or [13, Theorem 5] for different proofs. This inequality states that the sequence

$$(8) \quad k \mapsto \frac{\int_0^\infty x^k f_t(x) dx}{k!} \quad (k \geq 0)$$

is log-concave, completing the proof. ■

We now turn to the proof of the “only if” direction of Theorem 1.1, which relies on the Post inversion formula for the Laplace transform (see Feller [9, Section VII.6] or Widder [18, Section VII.1]). Suppose that φ is continuous on $[0, \infty)$ and C^∞ -smooth on $(0, \infty)$, and that the alternating Taylor coefficients $a_t(n)$ are log-concave. In particular, the alternating Taylor coefficients are non-negative. We use Bernstein’s theorem to conclude that there exists a finite, non-negative Borel measure μ on $[0, \infty)$ such that (2) holds true. All that remains is to prove the following:

PROPOSITION 2.2. *The measure μ is log-concave.*

The proof of Proposition 2.2 requires some preparation. First, it follows from (1) and (2) that for any $R, t > 0$,

$$(9) \quad \sum_{n=0}^{\lfloor Rt \rfloor} t^n a_t(n) = \sum_{n=0}^{\lfloor Rt \rfloor} \frac{t^n}{n!} \int_0^\infty x^n e^{-tx} d\mu(x) = \int_0^\infty \mathbb{P}(N_{tx} \leq Rt) d\mu(x),$$

where N_s is a Poisson random variable with parameter s , i.e.,

$$\mathbb{P}(N_s = n) = e^{-s} \frac{s^n}{n!}, \quad \text{for } n = 0, 1, 2, \dots$$

The random variable N_s has expectation s and standard deviation \sqrt{s} . By the central limit theorem for the Poisson distribution (see Feller [9,

Chapter VII] or Schilling, Song, and Vondraček [15, Lemma 1.1]), for any $\alpha > 0$,

$$(10) \quad \mathbb{P}(N_s \leq \alpha s) \xrightarrow{s \rightarrow \infty} \begin{cases} 1, & \alpha > 1, \\ 1/2, & \alpha = 1, \\ 0, & \alpha < 1. \end{cases}$$

The left-hand side of (10) is always between 0 and 1. Therefore we may use the bounded convergence theorem, and conclude from (9) that for any $R > 0$,

$$(11) \quad \lim_{t \rightarrow \infty} \sum_{n=0}^{\lfloor Rt \rfloor} t^n a_t(n) = \mu([0, R]) + \frac{1}{2}\mu(\{R\}).$$

For $t > 0$ define $g_t : [0, \infty) \rightarrow \mathbb{R}$ via

$$g_t(x) = \begin{cases} t^{n+1} \cdot a_t(n), & x = n/t \text{ for some integer } n \geq 0, \\ t^{x+1} \cdot a_t(n)^{1-\lambda} \cdot a_t(n+1)^\lambda, & x = (n+\lambda)/t \text{ for } \lambda \in (0, 1), n \geq 0. \end{cases}$$

Write μ_t for the measure on $[0, \infty)$ whose density is g_t . We think about μ_t as an approximation for the discrete measure on $[0, \infty)$ that has an atom at n/t of weight $t^n a_t(n)$ for any $n \geq 0$.

LEMMA 2.3. *Assume that $\mu((0, \infty)) > 0$. Then for any $t > 0$ the measure μ_t is log-concave on $[0, \infty)$. Moreover, if $\mu(\{0\}) = 0$ then for any $R > 0$,*

$$\mu_t([0, R]) \xrightarrow{t \rightarrow \infty} \mu([0, R]) + \frac{1}{2}\mu(\{R\}).$$

Proof. Since $\mu((0, \infty)) > 0$, for any $t > 0$ and $n \geq 0$,

$$a_t(n) = \int_0^\infty \frac{x^n}{n!} e^{-tx} d\mu(x) > 0.$$

The density g_t is locally Lipschitz, and for any integer $n \geq 0$ and $x \in (n/t, (n+1)/t)$,

$$(\log g_t)'(x) = \log t + t \log \frac{a_t(n+1)}{a_t(n)}.$$

The sequence $(a_t(n))_{n \geq 0}$ is log-concave, hence $a_t(n+1)/a_t(n)$ is non-increasing in n . We conclude that $(\log g_t)'(x)$, which exists for almost every $x > 0$, is a non-increasing function of $x \in [0, \infty)$. This shows that the locally Lipschitz function g_t is a log-concave function, and consequently μ_t is a log-concave measure. In particular, the density g_t is unimodal, meaning that for some $x_0 \geq 0$, the function g_t is non-decreasing in $(0, x_0)$ and non-increasing in (x_0, ∞) . We claim that for any $R, t > 0$ we have the Euler–Maclaurin type

bound

$$(12) \quad \left| \int_0^R g_t(x) dx - \sum_{n=0}^{\lfloor Rt \rfloor} \frac{1}{t} g_t\left(\frac{n}{t}\right) \right| \leq \frac{3}{t} \sup_{x>0} g_t(x).$$

Indeed, the sum in (12) is a Riemann sum related to the integral of g_t on the interval $I = [0, \lfloor tR + 1 \rfloor / t]$. This Riemann sum corresponds to a partition of I into segments of length $1/t$, and by unimodularity, this Riemann sum can deviate from the actual integral by at most $(2/t) \sup_{x>0} g_t(x)$. Since the symmetric difference between I and $[0, R]$ is an interval of length at most $1/t$, the relation (12) follows. According to (12), for any $R, t > 0$,

$$(13) \quad \left| \mu_t([0, R]) - \sum_{n=0}^{\lfloor Rt \rfloor} t^n a_t(n) \right| \leq \frac{3}{t} \sup_{x>0} g_t(x) = 3 \sup_{n \geq 0} t^n a_t(n).$$

Next we make use of our assumption that $\mu(\{0\}) = 0$ and also of the fact that $\sup_n e^{-s} s^n / n!$ tends to zero as $s \rightarrow \infty$, as may be verified routinely. This shows that for $t > 0$,

$$\sup_{n \geq 0} t^n a_t(n) = \sup_{n \geq 0} \int_0^\infty \frac{(tx)^n}{n!} e^{-tx} d\mu(x) \leq \int_0^\infty \left(\sup_{n \geq 0} \frac{(tx)^n}{n!} e^{-tx} \right) d\mu(x) \xrightarrow{t \rightarrow \infty} 0,$$

where we have used the dominated convergence theorem in the last passage. The lemma now follows from (11) and (13). ■

The following lemma is due to Borell, and its proof is contained in [3, Lemma 3.3] and the last paragraph of the proof of [3, Theorem 2.1]. For the reader's convenience, we include a short proof. For $A, B \subseteq \mathbb{R}$ and $\lambda \in \mathbb{R}$ we write $A + B = \{x + y; x \in A, y \in B\}$ and $\lambda A = \{\lambda x; x \in A\}$. We say that an interval $I \subseteq \mathbb{R}$ is *rational* if it has a finite length and if its endpoints are rational numbers.

LEMMA 2.4. *Let μ be a finite Borel measure on \mathbb{R} such that for any intervals $I, J \subseteq \mathbb{R}$ and $\lambda \in (0, 1)$,*

$$(14) \quad \mu(\lambda I + (1 - \lambda)J) \geq \mu(I)^\lambda \mu(J)^{1-\lambda}.$$

Then μ is log-concave (i.e., either $\mu = c\delta_{x_0}$ for some $c \geq 0, x_0 \in \mathbb{R}$ or else μ has a log-concave density). Moreover, the conclusion remains valid if we only assume that (14) holds for rational I, J and λ .

Proof. For $x \in \mathbb{R}$ and $\varepsilon > 0$ set

$$f_\varepsilon(x) = \frac{\mu((x - \varepsilon, x + \varepsilon))}{2\varepsilon}.$$

We deduce from (14) that $f_\varepsilon : \mathbb{R} \rightarrow (0, \infty)$ is a log-concave function for all $\varepsilon > 0$. Denote

$$f(x) = \limsup_{\varepsilon \rightarrow 0} f_\varepsilon(x) \in [0, +\infty] \quad \text{for } x \in \mathbb{R}.$$

Since f_ε is log-concave, it follows that for all $0 < \lambda < 1$ and $x, y \in \mathbb{R}$,

$$(15) \quad f(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda},$$

where in case $f(x) = \infty$ or $f(y) = \infty$ we interpret (15) by continuity. By the Lebesgue differentiation theorem, the function f is the density of the absolutely continuous component of the finite measure μ . In particular, f is integrable. Moreover, if $f(x) < \infty$ for all $x \in \mathbb{R}$, then the measure μ is absolutely continuous, and in any case, the set of all points $x \in \mathbb{R}$ where $f(x) = 0$ has zero μ -measure.

If $f(x) < \infty$ for all $x \in \mathbb{R}$, then μ is absolutely continuous with a log-concave density f , as required.

Otherwise, there exists $x_0 \in \mathbb{R}$ with $f(x_0) = +\infty$. Since $f(x_0) = +\infty$, necessarily $f(x) = 0$ for $x \neq x_0$, as otherwise (15) implies that f equals $+\infty$ in an interval of positive length, in contradiction to the integrability of f . Thus $f(x) = 0$ for all $x \neq x_0$, and μ is supported at the point $\{x_0\}$. In this case necessarily $\mu = c\delta_{x_0}$ for some $c \geq 0$.

For the second part of the lemma, observe that

$$\mu((x - \varepsilon, x + \varepsilon)) = \sup\{\mu(I); I \subsetneq (x - \varepsilon, x + \varepsilon) \text{ is a rational interval}\}.$$

Using this equality, one can show that if (14) holds for rational I, J and λ only, then f_ε is log-concave. We then proceed as above and conclude that μ is log-concave. ■

Proof of Proposition 2.2. We may assume that $\mu((0, \infty)) > 0$ as otherwise $\mu = c\delta_0$ and the conclusion trivially holds. Therefore $a_t(n) > 0$ for all t and n . By the log-concavity of the sequence of alternating Taylor coefficients,

$$(16) \quad \frac{\left(\int_0^\infty x e^{-tx} d\mu(x)\right)^2}{\int_0^\infty x^2 e^{-tx} d\mu(x)} = \frac{a_t(1)^2}{a_t(2)} \geq a_t(0) = \int_0^\infty e^{-tx} d\mu(x) \xrightarrow{t \rightarrow \infty} \mu(\{0\}).$$

For $t > 0$ write ν_t for the measure on $(0, \infty)$ whose density with respect to μ equals $x \mapsto e^{-tx}$. Then by the Cauchy–Schwarz inequality,

$$(17) \quad \frac{\left(\int_0^\infty x e^{-tx} d\mu(x)\right)^2}{\int_0^\infty x^2 e^{-tx} d\mu(x)} = \frac{\left(\int_0^\infty x d\nu_t(x)\right)^2}{\int_0^\infty x^2 d\nu_t(x)} \leq \nu_t((0, \infty)) \xrightarrow{t \rightarrow \infty} 0.$$

From (16) and (17) we see that $\mu(\{0\}) = 0$, which is required for the application of the second part of Lemma 2.3. Let $I, J \subseteq [0, \infty)$ be intervals and $0 < \lambda < 1$. Set $K = \lambda I + (1 - \lambda)J$ and assume first that

$$(18) \quad \mu(\partial I) = \mu(\partial J) = \mu(\partial K) = 0,$$

where ∂I is the boundary of the interval I . In this case, by Lemma 2.3,

$$(19) \quad \mu(I) = \lim_{t \rightarrow \infty} \mu_t(I), \quad \mu(J) = \lim_{t \rightarrow \infty} \mu_t(J), \quad \mu(K) = \lim_{t \rightarrow \infty} \mu_t(K).$$

Since μ_t is log-concave, the Prékopa–Leindler inequality (see, e.g., [6, Theorem 1.2.3]) implies that for all $t > 0$,

$$(20) \quad \mu_t(K) \geq \mu_t(I)^\lambda \mu_t(J)^{1-\lambda}.$$

From (19) and (20) we thus deduce that

$$(21) \quad \mu(\lambda I + (1 - \lambda)J) \geq \mu(I)^\lambda \mu(J)^{1-\lambda}.$$

We know that (21) holds true for any intervals $I, J \subseteq \mathbb{R}$ and $0 < \lambda < 1$ satisfying condition (18), where $K = \lambda I + (1 - \lambda)I$. Since μ is a finite measure, it can only have a countable number of atoms. Hence there exists $\alpha \in \mathbb{R}$ such that none of these atoms are congruent to $\alpha \pmod{\mathbb{Q}}$. In other words by translating μ , we may assume that it has no rational atoms. Then (21) holds for all rational I, J, λ and Lemma 2.4 implies that μ is a log-concave measure, as desired. ■

The proof of Theorem 1.1 is complete.

Proof of Corollary 1.2. If μ is of the form $c\delta_0$ for some $c \geq 0$, then the corollary is trivial. Otherwise, the alternating Taylor coefficients $a_t(n) = (-1)^n \varphi^{(n)}(t)/n!$ are positive for every $t > 0$ and $n \geq 0$. The measure μ is log-concave if and only if the sequence of alternating Taylor coefficients is log-concave for any $t > 0$, which happens if and only if

$$(22) \quad \left(\frac{\varphi^{(n)}(t)}{n!} \right)^2 - \frac{\varphi^{(n-1)}(t)}{(n-1)!} \cdot \frac{\varphi^{(n+1)}(t)}{(n+1)!} \geq 0 \quad (n \geq 1, t > 0).$$

Denote by $b_n(t)$ the expression on the left-hand side of (22) multiplied by $(n!)^2$. Then,

$$\begin{aligned} \frac{d^2}{dt^2} |\varphi^{(n-1)}(t)|^{-1/n} &= \frac{d^2}{dt^2} ((-1)^{n-1} \cdot \varphi^{(n-1)}(t))^{-1/n} \\ &= \frac{n+1}{n^2} |\varphi^{(n-1)}(t)|^{-(2n+1)/n} \cdot b_n(t). \end{aligned}$$

Hence $b_n(t) \geq 0$ for all $t > 0$ if and only if the function $|\varphi^{(n-1)}(t)|^{-1/n}$ is convex in $(0, \infty)$. ■

3. The log-concavity measurements are completely monotone.

In this section we prove Proposition 1.3 and Theorem 1.5. Let $k \leq \ell \leq m \leq n$ be non-negative integers with $k + n = \ell + m$ and let φ be a continuous function on $[0, \infty)$ that is C^∞ -smooth in $(0, \infty)$. Define

$$\begin{aligned} c_{k,\ell,m,n}(t) &= a_t(\ell)a_t(m) - a_t(k)a_t(n) \\ &= (-1)^{\ell+m} \left[\frac{\varphi^{(\ell)}(t)}{\ell!} \frac{\varphi^{(m)}(t)}{m!} - \frac{\varphi^{(k)}(t)}{k!} \frac{\varphi^{(n)}(t)}{n!} \right], \end{aligned}$$

where $a_t(n) = (-1)^n \varphi^{(n)}(t)/n!$ as before. We call the functions $c_{k,\ell,m,n} : (0, \infty) \rightarrow \mathbb{R}$ the *log-concavity measurements* of φ . This name is justified by the following minor lemma. For integers $a \leq b$ we write $\llbracket a, b \rrbracket = \{n \in \mathbb{Z}; a \leq n \leq b\}$.

LEMMA 3.1. *Let $t > 0$. Then the sequence $(a_t(n))_{n \geq 0}$ is log-concave if and only if $c_{k,\ell,m,n}(t) \geq 0$ for all non-negative integers $k \leq \ell \leq m \leq n$ with $k + n = \ell + m$.*

Proof. Assume first that the sequence $(a_t(n))_{n \geq 0}$ is log-concave. Fix non-negative integers $k \leq \ell \leq m \leq n$ with $k + n = \ell + m$, and let us prove that $c_{k,\ell,m,n}(t) \geq 0$. This is obvious in the case where $k = \ell$ and $m = n$. Otherwise, set $\lambda = (\ell - k)/(n - k) \in (0, 1)$. Then $\ell = \lambda n + (1 - \lambda)k$ and $m = \lambda k + (1 - \lambda)n$. According to (3),

$$a_t(\ell) \geq a_t(k)^{1-\lambda} a_t(n)^\lambda \quad \text{and} \quad a_t(m) \geq a_t(k)^\lambda a_t(n)^{1-\lambda}.$$

By multiplying these two inequalities, we conclude that $c_{k,\ell,m,n}(t) \geq 0$. For the other direction, assume that the log-concavity measurements are non-negative. In particular, for any $n \geq 1$,

$$0 \leq c_{n-1,n,n,n+1}(t) = a_t(n)^2 - a_t(n-1)a_t(n+1).$$

It remains to show that the set of non-negative integers n with $a_t(n) > 0$ is an interval of integers. Assume that $k \leq n$ satisfy $a_t(k) > 0$ and $a_t(n) > 0$. Given any integer $\ell \in \llbracket k, n \rrbracket$, we set $m = n + k - \ell \in \llbracket k, n \rrbracket$. By the non-negativity of the log-concavity measurements,

$$(23) \quad a_t(\ell)a_t(m) \geq a_t(k)a_t(n) > 0.$$

From (23) we deduce that $a_t(\ell) > 0$ for any integer $\ell \in \llbracket k, n \rrbracket$, as desired. ■

PROPOSITION 3.2. *The derivative of each log-concavity measurement is a linear combination with constant, non-positive coefficients of a finite number of log-concavity measurements.*

Proof. Differentiating (1) we obtain

$$\frac{d}{dt} a_t(n) = -(n+1)a_t(n+1) \quad (t > 0, n \geq 0).$$

Abbreviate $b_j = a_t(j)$. Then

$$(24) \quad -c'_{k,\ell,m,n}(t) = (\ell+1)b_{\ell+1}b_m + (m+1)b_\ell b_{m+1} - (k+1)b_{k+1}b_n - (n+1)b_k b_{n+1}.$$

Assume first that $\ell < m$. In this case we may rewrite the right-hand side of (24) as

$$(k+1)[b_{\ell+1}b_m - b_{k+1}b_n] + (\ell-k)[b_{\ell+1}b_m - b_k b_{n+1}] + (m+1)[b_\ell b_{m+1} - b_k b_{n+1}].$$

Therefore, in the case $\ell < m$, we expressed $-c'_{k,\ell,m,n}(t)$ as a linear combination with non-negative coefficients of three log-concavity measurements.

From now on, we consider the case $\ell = m$. If $k = \ell$, then necessarily $n = m$ and the log-concavity measurement $c_{k,\ell,m,n}(t)$ vanishes. If $k < \ell$, then necessarily $m < n$ and we rewrite the right-hand side of (24) as

$$(k + 1)[b_\ell b_{m+1} - b_{k+1} b_n] + (n + 1)[b_\ell b_{m+1} - b_k b_{n+1}].$$

Consequently, in the case $\ell = m$, we may express $-c'_{k,\ell,m,n}(t)$ as a linear combination of two log-concavity measurements with non-negative coefficients. The proof is complete. ■

COROLLARY 3.3. *Let $t > 0$ be such that $(a_t(n))_{n \geq 0}$ is a log-concave sequence. Assume that $k \leq \ell \leq m \leq n$ are non-negative integers with $k + n = \ell + m$. Abbreviate $f(t) = c_{k,\ell,m,n}(t)$. Then for all $j \geq 0$,*

$$(-1)^j f^{(j)}(t) \geq 0.$$

Proof. Any log-concavity measurement is non-negative at any $t > 0$. It follows from Proposition 3.2 that $(-1)^j f^{(j)}(t)$ is a finite linear combination of log-concavity measurements with non-negative coefficients. Therefore $(-1)^j f^{(j)}(t) \geq 0$. ■

Proof of Proposition 1.3. Write $A \subseteq (0, \infty)$ for the set of all $t > 0$ for which $(a_t(n))_{n \geq 0}$ is a log-concave sequence. Since φ is C^∞ -smooth, the set A is closed in $(0, \infty)$. From our assumption, $s \in A$. Define

$$t_0 = \inf\{t > 0; [t, s] \subseteq A\}.$$

Then $t_0 \leq s$. Our goal is to prove that $t_0 = 0$. Assume by contradiction that $t_0 > 0$. Since A is a closed set, necessarily $t_0 \in A$. Since φ is real-analytic, the Taylor series of φ converges to φ in $(t_0 - \varepsilon, t_0 + \varepsilon)$ for a certain $\varepsilon > 0$. Assume that $k \leq \ell \leq m \leq n$ are non-negative integers with $k + n = \ell + m$. Then also the Taylor series of $f(t) = c_{k,\ell,m,n}(t)$ converges to f in the same interval $(t_0 - \varepsilon, t_0 + \varepsilon)$. From Corollary 3.3 we thus deduce that for all $t \in (t_0 - \varepsilon, t_0]$,

$$c_{k,\ell,m,n}(t) \geq 0.$$

Consequently, $(t_0 - \varepsilon, t_0] \subseteq A$, in contradiction to the definition of t_0 . ■

We proceed with yet another proof of Proposition 1.3, which is more in the spirit of the Walkup theorem which we shall now recall:

THEOREM 3.4 (Walkup theorem [13, 17]). *If $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ are log-concave sequences, then the sequence $(c_n)_{n \geq 0}$ given by*

$$c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k} \quad (n \geq 0)$$

is also log-concave.

By Taylor’s theorem, whenever $0 < s < t$,

$$(t - s)^k a_s(k) = \sum_{n=k}^{\infty} \binom{n}{k} (t - s)^n a_t(n),$$

assuming that φ is real-analytic and that the Taylor series of φ at t converges in $(s - \varepsilon, t + \varepsilon)$ for some $\varepsilon > 0$. We conclude that Proposition 1.3 is equivalent to the following Walkup-type result:

PROPOSITION 3.5. *If $(a_k)_{k \geq 0}$ is a log-concave sequence then the sequence $(c_k)_{k \geq 0}$ defined by*

$$c_k = \sum_{n=k}^{\infty} \binom{n}{k} a_n \quad (k \geq 0)$$

is log-concave as well.

We do not know of a formal derivation of Theorem 3.4 from Proposition 3.5 or vice versa, yet we provide a direct proof of Proposition 3.5 which bears some similarity to the proof of Walkup’s theorem and the Borell–Berwald inequality given in [13, 17]. We begin the direct proof of Proposition 3.5 with the following:

LEMMA 3.6. *Let $(a_n)_{n \geq 0}$ be a log-concave sequence. Then for every non-negative integers k and l we have*

$$(25) \quad \sum_{n \geq 0} \binom{n}{k} \binom{l-n}{k} a_n a_{l-n} \geq \sum_{n \geq 0} \binom{n}{k-1} \binom{l-n}{k+1} a_n a_{l-n},$$

where we set $\binom{n}{k} = 0$ when $k > n$ or $k < 0$ or $n < 0$.

Proof. Inequality (25) holds trivially if $2k > l$. We may thus assume that $2k \leq l$. Let U be a random subset of cardinality $2k + 1$ of $\{1, \dots, l + 1\}$ chosen uniformly. Let X_1, \dots, X_{2k+1} be the elements of U in increasing order. Observe that the law of X_{k+1} is given by

$$\mathbb{P}(X_{k+1} = n + 1) = \frac{\binom{n}{k} \binom{l-n}{k}}{\binom{l+1}{2k+1}} \quad (n \geq 0).$$

Therefore

$$\sum_n \binom{n}{k} \binom{l-n}{k} a_n a_{l-n} = \binom{l+1}{2k+1} \mathbb{E}[f(X_{k+1})]$$

where f is the function given by

$$f(n) = a_{n-1} a_{l+1-n}, \quad \forall n,$$

and we set $a_k = 0$ for $k < 0$. In a similar way

$$\sum_n \binom{n}{k-1} \binom{l-n}{k+1} a_n a_{l-n} = \binom{l+1}{2k+1} \mathbb{E}[f(X_k)].$$

Hence the desired inequality boils down to

$$\mathbb{E}[f(X_k)] \leq \mathbb{E}[f(X_{k+1})].$$

By Fubini it suffices to prove that

$$\mathbb{P}(f(X_k) > t) \leq \mathbb{P}(f(X_{k+1}) > t), \quad \forall t \geq 0.$$

The sequence $(f(n))_{n \geq 0}$ is log-concave, since it is the pointwise product of two log-concave sequences. It also satisfies $f(l+2-n) = f(n)$ for all n . The crucial observation is that because of the log-concavity and symmetry of the sequence $(f(n))_{n \geq 0}$, the farther n is from the midpoint $(l+2)/2$, the smaller is $f(n)$. Therefore the level set $\{f > t\}$ is either empty or an interval of the form $\llbracket n, l+2-n \rrbracket$ for some integer $n \leq (l+2)/2$. Hence it suffices to prove that for every such n ,

$$(26) \quad \mathbb{P}(X_k \in \llbracket n, l+2-n \rrbracket) \leq \mathbb{P}(X_{k+1} \in \llbracket n, l+2-n \rrbracket).$$

Intuitively, since X_{k+1} is the middle element of U , it is more likely to be close to the center of the interval $\llbracket 1, l+1 \rrbracket$ than any other element. More precisely, since $X_k \leq X_{k+1}$,

$$\begin{aligned} & \mathbb{P}(X_k \in \llbracket n, l+2-n \rrbracket) - \mathbb{P}(X_{k+1} \in \llbracket n, l+2-n \rrbracket) \\ &= \mathbb{P}(X_k \leq l+2-n; X_{k+1} > l+2-n) - \mathbb{P}(X_k < n; X_{k+1} \geq n) \\ &= \frac{\binom{l+2-n}{k} \binom{n-1}{k+1}}{\binom{l+1}{2k+1}} - \frac{\binom{n-1}{k} \binom{l+2-n}{k+1}}{\binom{l+1}{2k+1}}. \end{aligned}$$

In order to complete the proof of (26) we need to show that this expression is non-positive, assuming that $k \leq l/2$ and $n \leq (l+2)/2$. Note that

$$(27) \quad \frac{\binom{l+2-n}{k} \binom{n-1}{k+1}}{\binom{n-1}{k} \binom{l+2-n}{k+1}} = \frac{(n-1-k)!(l-n-k+1)!}{(l+2-n-k)!(n-k-2)!} = \frac{(n-1)-k}{(l+2-n)-k}.$$

We need to show that the expression in (27) is at most one. The denominator in (27) is positive, as

$$l+2-n-k = 1 + \llbracket (l+2)/2 - n \rrbracket + (l/2 - k) \geq 1.$$

The numerator in (27) is smaller than the denominator, as $n-1 < l+2-n$. Hence the expression in (27) is at most one, completing the proof of the lemma. ■

Direct proof of Proposition 3.5. The set of all k with $c_k > 0$ is clearly the interval of integers $\{k \geq 0; \exists n \geq k, a_n > 0\}$. Let $k \geq 0$ be an integer. We need to prove that

$$c_k^2 = \sum_{n,m} \binom{n}{k} \binom{m}{k} a_n a_m \geq \sum_{n,m} \binom{n}{k-1} \binom{m}{k+1} a_n a_m = c_{k-1} c_{k+1}.$$

By grouping the terms according to the value of $n + m$ we see that it suffices to prove that for any $l, k \geq 0$,

$$\sum_n \binom{n}{k} \binom{l-n}{k} a_n a_{l-n} \geq \sum_n \binom{n}{k-1} \binom{l-n}{k+1} a_n a_{l-n}.$$

This is, however, precisely the statement of the previous lemma. ■

When μ is a finite, log-concave measure on $[0, \infty)$, it is well-known (e.g., [6]) that $\mu([t, \infty)) \leq Ae^{-Bt}$ for all $t > 0$, where $A, B > 0$ depend only on μ . It follows that the Laplace transform φ defined in (2) is holomorphic in $\{t \in \mathbb{C}; \operatorname{Re}(t) > -B\}$ for some $B > 0$ depending on μ .

Proof of Theorem 1.5. By Theorem 1.1, the alternating Taylor coefficients sequence $(a_t(n))_{n \geq 0}$ is log-concave for any $t > 0$. From Corollary 3.3 we thus learn that

$$f(t) = c_{k,\ell,m,n}(t) = a_t(\ell)a_t(m) - a_t(k)a_t(n)$$

satisfies $(-1)^j f^{(j)}(t) \geq 0$ for any $t > 0$ and $j \geq 0$. The function f is real-analytic in a neighborhood of $[0, \infty)$ and in particular it is continuous in $[0, \infty)$. The function f is thus completely monotone, and according to the Bernstein theorem, there exists a finite, non-negative measure ν for which (6) holds true. ■

We may rewrite conclusion (6) of Theorem 1.5 as follows: For any $t > 0$,

$$\begin{aligned} \int_0^\infty \frac{x^\ell}{\ell!} e^{-tx} d\mu(x) \int_0^\infty \frac{x^m}{m!} e^{-tx} d\mu(x) - \int_0^\infty \frac{x^k}{k!} e^{-tx} d\mu(x) \int_0^\infty \frac{x^n}{n!} e^{-tx} d\mu(x) \\ = \int_0^\infty e^{-tx} d\nu(x). \end{aligned}$$

Let us now consider the Fourier transform

$$F_\mu(t) = \int_0^\infty e^{-itx} d\mu(x) \quad (t \in \mathbb{R}).$$

By analytic continuation, Theorem 1.5 immediately implies the following:

PROPOSITION 3.7. *Let μ be a finite, log-concave measure on $[0, \infty)$. Then for any non-negative integers $k \leq \ell \leq m \leq n$ with $k + n = \ell + m$ there exists a finite, non-negative measure $\nu = \nu_{k,\ell,m,n}$ on $[0, \infty)$ such that for any $t > 0$,*

$$\frac{F_\mu^{(\ell)}(t)}{\ell!} \frac{F_\mu^{(m)}(t)}{m!} - \frac{F_\mu^{(k)}(t)}{k!} \frac{F_\mu^{(n)}(t)}{n!} = (-i)^{\ell+m} F_\nu(t).$$

COROLLARY 3.8. *Let μ, k, ℓ, m, n, ν be as in Theorem 1.5. Write $P_j(\mu)$ for the measure whose density with respect to μ is $x \mapsto x^j/j!$. Write $E_t(\mu)$ for the measure whose density with respect to μ is $x \mapsto \exp(-tx)$. Then:*

- (i) We have $\nu = P_\ell(\mu) * P_m(\mu) - P_k(\mu) * P_n(\mu)$ where $*$ stands for convolution.
- (ii) For any $t > 0$, the measure $E_t(\nu)$ is the Berwald–Borell transform of $E_t(\mu)$ with the same parameters (k, ℓ, m, n) . The same holds for any $t \in \mathbb{R}$ for which $E_t(\mu)$ is a finite measure.

Proof. We note that $F_\mu^{(j)}/j! = (-i)^j F_{P_j(\mu)}$. Proposition 3.7 thus shows that

$$(28) \quad F_{P_\ell(\mu)} F_{P_m(\mu)} - F_{P_k(\mu)} F_{P_n(\mu)} = F_\nu.$$

The Fourier transform maps products to convolutions. Conclusion (i) therefore follows from (28). Conclusion (ii) follows immediately from the definitions. ■

4. Borell-type formula for the Poisson measure. In [4], Borell gave a new proof of the Prékopa–Leindler inequality based on the following stochastic variational formula. Let γ_n be the standard Gaussian measure on \mathbb{R}^n . Given a standard n -dimensional Brownian motion $(B_t)_{t \geq 0}$ and a bounded, measurable function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ we have

$$(29) \quad \log \left(\int_{\mathbb{R}^n} e^f d\gamma_n \right) = \sup_u \left\{ \mathbb{E} \left[f \left(B_1 + \int_0^1 u_s ds \right) - \frac{1}{2} \int_0^1 |u_s|^2 ds \right] \right\},$$

where the supremum is taken over all bounded stochastic processes u which are adapted to the Brownian filtration, i.e. u_t is measurable with respect to the σ -field generated by $\{B_s; s \leq t\}$ for all $t \in [0, 1]$.

In this section we give a discrete version of Borell’s formula in which the Gaussian measure and the Brownian motion are replaced by the Poisson distribution and the Poisson process, respectively. In the following section we shall apply our formula in order to deduce a discrete version of the Prékopa–Leindler inequality. We begin with some background on counting processes with stochastic intensities. Let $T > 0$ be a fixed number, denote $\mathbb{R}_+ = [0, \infty)$, and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which our random variables will be defined.

Throughout this section, we let N be a Poisson point process on $[0, T] \times \mathbb{R}_+ \subseteq \mathbb{R}^2$ with intensity measure equal to the Lebesgue measure \mathcal{L} . In particular $N(F)$ is a Poisson random variable with parameter $\mathcal{L}(F)$ for any Borel set $F \subseteq [0, T] \times \mathbb{R}_+$. For a Borel subset $E \subseteq [0, T] \times \mathbb{R}_+$ we write \mathcal{F}_E for the σ -field generated by the random variables

$$\{N(F); F \text{ is a Borel set, } F \subseteq E\}.$$

For $t \in [0, T]$ we set $\mathcal{F}_t = \mathcal{F}_{[0, t] \times \mathbb{R}_+}$. This defines a filtration of Ω . Recall that a stochastic process $(\lambda_t)_{0 \leq t \leq T}$ is called *predictable* if, as a function of $t \in [0, T]$ and $\omega \in \Omega$, it is measurable with respect to the σ -field \mathcal{P} generated

by the sets

$$\{(s, t] \times A; s \leq t \leq T, A \in \mathcal{F}_s\}.$$

This is slightly more restrictive than being adapted, i.e., when λ_t is measurable with respect to \mathcal{F}_t . We have the following standard fact: if a process is left-continuous and adapted, then it is predictable.

Given a predictable, bounded, non-negative stochastic process $(\lambda_t)_{0 \leq t \leq T}$ we define the associated counting process $(X_t^\lambda)_{0 \leq t \leq T}$ via

$$(30) \quad X_t^\lambda = N(\{(s, u) \in [0, T] \times \mathbb{R}^+; s < t, u \leq \lambda_s\}).$$

In other words, X_t^λ is the number of atoms of N which lie below the curve $\{(s, \lambda_s); s \in [0, t]\}$. The counting process X^λ defined via (30) is clearly adapted, non-decreasing, integer-valued and left-continuous. Note that given $M > 0$, with probability 1 the process N has only finitely many atoms in the box $[0, T] \times [0, M]$ and no two of those lie on the same vertical line $\{t\} \times [0, M]$. Thus, with probability one the process X^λ has finitely many jumps, all of size 1. We sometimes refer to (λ_t) as the *stochastic intensity* of the counting process (X_t^λ) , and to the jumps of X^λ as *atoms*.

LEMMA 4.1. *For every non-negative predictable process $(H_t)_{0 \leq t \leq T}$ we have*

$$(31) \quad \mathbb{E} \left[\int_0^T H_t X^\lambda(dt) \right] = \mathbb{E} \left[\int_0^T H_t \lambda_t dt \right],$$

where the integral on the left-hand side is a Riemann–Stieltjes integral, i.e., here it is a sum of the values of H_t at the atoms of X^λ .

The proof of the technical Lemma 4.1 is deferred to the appendix. Equation (31) is frequently taken as the definition of a counting process with stochastic intensity λ . The process

$$\tilde{X}_t^\lambda = X_t^\lambda - \int_0^t \lambda_s ds \quad (0 \leq t \leq T)$$

is called the *compensated* process. By Lemma 4.1 it has the property that for every bounded, predictable process $(H_t)_{0 \leq t \leq T}$, the process

$$\left(\int_0^t H_s \tilde{X}^\lambda(ds) \right)_{0 \leq t \leq T}$$

is a martingale. We are now in a position to state the analogue of Borell's formula for the Poisson measure. In the following theorem π_T denotes the Poisson measure with parameter T , i.e.,

$$\pi_T(n) = \frac{T^n}{n!} e^{-T} \quad \text{for } n \in \mathbb{N} = \{0, 1, 2, \dots\}$$

where we abbreviate $\pi_T(n) = \pi_T(\{n\})$.

THEOREM 4.2. Let $f: \mathbb{N} \rightarrow \mathbb{R}$ be bounded and let $T > 0$. Then

$$(32) \quad \log \left(\int_{\mathbb{N}} e^f d\pi_T \right) = \sup_{\lambda} \left\{ \mathbb{E} \left[f(X_T^\lambda) - \int_0^T (\lambda_t \log \lambda_t - \lambda_t + 1) dt \right] \right\},$$

where the supremum is taken over all bounded, non-negative, predictable processes $(\lambda_t)_{0 \leq t \leq T}$, and $(X_t^\lambda)_{0 \leq t \leq T}$ is the associated counting process, defined by (30). Moreover the supremum is actually a maximum.

Proof. Let $(P_t)_{t \geq 0}$ be the Poisson semigroup: For every $g: \mathbb{N} \rightarrow \mathbb{R}$,

$$P_t g(x) = \sum_{n \in \mathbb{N}} g(x+n) \pi_t(n).$$

We shall show that for every predictable, non-negative, bounded process (λ_t) we have

$$(33) \quad \log P_T(e^f)(0) \geq \mathbb{E} \left[f(X_T^\lambda) - \int_0^T (\lambda_t \log \lambda_t - \lambda_t + 1) dt \right],$$

with equality if λ is chosen appropriately. Let us start with the inequality. Note that for every $g: \mathbb{N} \rightarrow \mathbb{R}$ and $t \geq 0$,

$$\partial_t P_t g = \partial_x P_t g$$

where $\partial_x g(x) = g(x+1) - g(x)$ denotes the discrete gradient. Letting

$$F(t, x) = \log P_{T-t}(e^f)(x)$$

we obtain

$$\partial_t F = -e^{\partial_x F} + 1.$$

Let λ be a predictable, non-negative, bounded process and let

$$M_t = F(t, X_t^\lambda) - \int_0^t (\lambda_s \log \lambda_s - \lambda_s + 1) ds.$$

Almost surely, the process $(M_t)_{0 \leq t \leq T}$ is a piecewise absolutely continuous function in t . Thus the distributional derivative of the function $t \mapsto M_t$ is almost surely the sum of an integrable function on $[0, T]$ and finitely many atoms. Namely, for any fixed $t \in [0, T]$,

$$(34) \quad M_t - M_0 = \int_0^t \partial_x F(s, X_s^\lambda) X^\lambda(ds) - \int_0^t (e^{\partial_x F(s, X_s^\lambda)} + \lambda_s \log \lambda_s - \lambda_s) ds.$$

Setting $\alpha_t = \partial_x F(t, X_t^\lambda)$, we may rewrite (34) as follows:

$$(35) \quad M_t - M_0 = \int_0^t \alpha_s \tilde{X}^\lambda(ds) - \int_0^t (e^{\alpha_s} + \lambda_s \log \lambda_s - \lambda_s - \alpha_s \lambda_s) ds.$$

Recall that $\widetilde{X}_t^\lambda = X_t^\lambda - \int_0^t \lambda_s ds$ is the compensated process. Note that $F(t, x)$ is continuous in t and that (X_t^λ) is left-continuous in t . Thus (α_t) is left-continuous. Since (α_t) is also adapted, it is predictable. Moreover both (α_t) and (λ_t) are bounded. Consequently, the first summand on the right-hand side of (35) is a martingale. Furthermore, since

$$(36) \quad e^x + y \log y - y - xy \geq 0, \quad \forall x \in \mathbb{R}, y \in \mathbb{R}_+,$$

the second integral on the right-hand side of (35) is non-negative. Therefore $(M_t)_{0 \leq t \leq T}$ is a supermartingale. In particular $M_0 \geq \mathbb{E}[M_T]$, which is the desired inequality (33).

There is equality in (36) if $e^x = y$. Hence if λ is such that

$$(37) \quad \lambda_t = e^{\partial_x F(t, X_t^\lambda)},$$

for almost every t and with probability 1, then M is a martingale and we have equality in (33). Note that the function $e^{\partial_x F(t, x)}$ is continuous in t and bounded. In Lemma 4.3 below we prove that under these conditions, a solution to (37) does indeed exist, which concludes the proof of the theorem. ■

REMARKS. 1. It is also possible to prove Theorem 4.2 by using the Girsanov change of measure formula for counting processes. The argument presented here has the advantage of being self-contained.

2. Theorem 4.2 can be generalized in several ways. Firstly, up to some technical details, the argument should work just the same for a function f that depends on the whole trajectory of the process rather than just the terminal point. On the left-hand side, the Poisson distribution should then be replaced by the law of the Poisson process of intensity 1 on $[0, T]$. In the Gaussian case, this pathspace version of the formula is known as the Boué–Dupuis formula [5]. Then one can also replace the interval $[0, T]$ equipped with the Lebesgue measure by a more general measure space, leading to a Borell-type formula for Poisson point processes. This program was actually already carried out by Budhiraja, Dupuis and Maroulas [7]. Theorem 4.2 is thus a particular case of their main result. However, their argument is a lot more intricate than the above proof.

3. A dual version of Borell’s formula involving relative entropy was proved by the second-named author [12]. This can be done in the Poisson case too. The formula then reads: If μ is a probability measure on \mathbb{N} whose density with respect to π_T is bounded away from 0 and $+\infty$, then the relative entropy of μ with respect to π_T satisfies

$$H(\mu \mid \pi_T) = \inf_{\lambda} \left\{ \mathbb{E} \left[\int_0^T (\lambda_t \log \lambda_t - \lambda_t + 1) dt \right] \right\},$$

where the infimum is taken over all non-negative, bounded, predictable processes λ such that X_T^λ has law μ . This follows from the representation for-

mula (32) and the Gibbs variational principle, which is the fact that the functionals $\nu \mapsto H(\nu|\pi_T)$ and $f \mapsto \log \int e^f d\pi_T$ are Legendre–Fenchel conjugates with respect to the pairing $(f, \nu) \mapsto \int f d\nu$.

We now state and prove the technical lemma used in the proof of Theorem 4.2.

LEMMA 4.3. *Let $G: [0, T] \times \mathbb{N} \rightarrow \mathbb{R}_+$ and assume that G is continuous in the first variable and bounded. Then there exists a predictable, bounded, non-negative process $(\lambda_t)_{0 \leq t \leq T}$ satisfying*

$$\lambda_t = G(t, X_t^\lambda)$$

for almost every $t \leq T$ and with probability 1.

Proof. Consider the map

$$H: (\lambda_t)_{0 \leq t \leq T} \mapsto (G(t, X_t^\lambda))_{0 \leq t \leq T}$$

from the set of predictable, non-negative processes to itself. We will show that H has a fixed point. Let λ and μ be two processes in the domain of H . Since G is bounded, there is a constant $C > 0$ such that

$$\mathbb{E}[|G(t, X_t^\lambda) - G(t, X_t^\mu)|] \leq C \mathbb{P}(X_t^\lambda \neq X_t^\mu).$$

The probability that the integer-valued random variable X_t^λ differs from X_t^μ is dominated by $\mathbb{E}[|X_t^\lambda - X_t^\mu|]$, which in turn is the average number of atoms of N between the graphs of λ and of μ on $[0, t)$. Since λ and μ are predictable, it follows from Lemma 4.1 that

$$\mathbb{E}|X_t^\lambda - X_t^\mu| = \mathbb{E}\left[\int_0^t |\lambda_s - \mu_s| ds\right].$$

Therefore

$$\mathbb{E}[|G(t, X_t^\lambda) - G(t, X_t^\mu)|] \leq C \mathbb{E}\left[\int_0^t |\lambda_s - \mu_s| ds\right].$$

This easily implies that H is Lipschitz with constant 1/2 for the distance

$$d(\lambda, \mu) = \int_0^T e^{-2Ct} \mathbb{E}[|\lambda_t - \mu_t|] dt.$$

Note that $d(\lambda, \mu) = 0$ if and only if $\lambda_t = \mu_t$ for almost every $0 \leq t \leq T$ and with probability one. The space of all predictable, non-negative processes $(\lambda_t)_{0 \leq t \leq T}$ with $\int_0^T \mathbb{E}|\lambda_t| dt < \infty$ is complete with respect to the distance d . Thus, being a contraction, the map H has a fixed point. This fixed point is necessarily a bounded process, since G is bounded. This completes the proof. ■

5. A discrete Prékopa–Leindler inequality. Following Borell, in this section we derive a Prékopa–Leindler type inequality from the representation formula (32). Recall that if x is a real number we denote its lower integer part by $\lfloor x \rfloor$ and its upper integer part by $\lceil x \rceil$. For $a, b \in \mathbb{R}$ we denote $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Recall that π_T denotes the Poisson distribution with parameter T .

PROPOSITION 5.1. *Let $T > 0$ and let $f, g, h, k: \mathbb{N} \rightarrow \mathbb{R}$ satisfy*

$$f(x) + g(y) \leq h\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right) + k\left(\left\lceil \frac{x+y}{2} \right\rceil\right), \quad \forall x, y \in \mathbb{N}.$$

Then,

$$\int_{\mathbb{N}} e^f d\pi_T \int_{\mathbb{N}} e^g d\pi_T \leq \int_{\mathbb{N}} e^h d\pi_T \int_{\mathbb{N}} e^k d\pi_T.$$

Proof. By approximation we may assume that all four functions are bounded. Let α and β be two non-negative, bounded, predictable processes. It follows from formula (30) that $\lfloor (X^\alpha + X^\beta)/2 \rfloor$ coincides with the process X^λ , where

$$(38) \quad \lambda = (\alpha \wedge \beta)\chi + (\alpha \vee \beta)(1 - \chi),$$

and χ_t is the indicator function of the event that $X_t^\alpha + X_t^\beta$ is even. Indeed, we see from the definition (30) that an atom of the Poisson process N that lies below the graph of $\alpha \wedge \beta$ corresponds to a jump both in X^α and in X^β , and consequently it entails a jump in $\lfloor (X^\alpha + X^\beta)/2 \rfloor$. On the other hand, an atom of N that lies between the graphs of $\alpha \wedge \beta$ and $\alpha \vee \beta$ causes a jump only if $\chi_t = 0$. This explains formula (38). Since X^α and X^β are adapted and left-continuous in t , the same applies to χ . Consequently, χ and λ are predictable.

Similarly $\lceil (X^\alpha + X^\beta)/2 \rceil = X^\mu$, where

$$\mu = (\alpha \wedge \beta)(1 - \chi) + (\alpha \vee \beta)\chi.$$

Note that for every $t \in [0, T]$ either $\mu_t = \alpha_t$ and $\lambda_t = \beta_t$ or the other way around. In particular, for every function $\varphi: [0, \infty) \rightarrow \mathbb{R}$,

$$\varphi(\alpha_t) + \varphi(\beta_t) = \varphi(\lambda_t) + \varphi(\mu_t), \quad \forall t \in [0, T].$$

Using the hypothesis made on f, g, h, k we deduce that for a continuous function φ ,

$$\begin{aligned} f(X_T^\alpha) + g(X_T^\beta) - \int_0^T \varphi(\alpha_t) dt - \int_0^T \varphi(\beta_t) dt \\ \leq h(X_T^\lambda) + k(X_T^\mu) - \int_0^T \varphi(\lambda_t) dt - \int_0^T \varphi(\mu_t) dt. \end{aligned}$$

Choosing $\varphi(x) = x \log x + x - 1$, taking expectation, and using the representation formula in Theorem 4.2 for h and k we obtain

$$\begin{aligned} \mathbb{E} \left[f(X_T^\alpha) - \int_0^T \varphi(\alpha_t) dt \right] + \mathbb{E} \left[g(X_T^\beta) - \int_0^T \varphi(\beta_t) dt \right] \\ \leq \log \left(\int_{\mathbb{N}} e^h d\pi_T \right) + \log \left(\int_{\mathbb{N}} e^k d\pi_T \right). \end{aligned}$$

Taking the supremum in α and β and using the representation formula for f and g yields the result. ■

Rescaling appropriately, we obtain as a corollary a Prékopa–Leindler type inequality for the counting measure on \mathbb{Z} .

Proof of Theorem 1.4. We may assume that all four sums in (4) are finite. Let Y_n be a random variable having the Poisson law with parameter n and let $X_n = Y_n - n$. Applying the previous proposition to the functions f, g, h, k (translated by $-n$) we get

$$(39) \quad \mathbb{E}[e^{f(X_n)}] \mathbb{E}[e^{g(X_n)}] \leq \mathbb{E}[e^{h(X_n)}] \mathbb{E}[e^{k(X_n)}].$$

On the other hand, for any fixed $k \in \mathbb{Z}$, letting n tend to $+\infty$ and using the Stirling formula we get

$$\mathbb{P}(X_n = k) = \frac{n^{n+k}}{(n+k)!} e^{-n} = \frac{1}{\sqrt{2\pi n}} (1 + o(1)).$$

Hence by the dominated convergence theorem

$$\sqrt{2\pi n} \cdot \mathbb{E}[e^{f(X_n)}] \xrightarrow{n \rightarrow \infty} \sum_{x \in \mathbb{Z}} e^{f(x)},$$

and similarly for g, h, k . Therefore multiplying (39) by n and letting n tend to $+\infty$ yields the result. ■

6. Appendix: Proof of Lemma 4.1. We write $\mathcal{B}([0, T])$ for the Borel σ -field of $[0, T]$. Let μ^+ and μ^- be the measures on $[0, T] \times \mathbb{R}_+ \times \Omega$ equipped with the σ -field $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ defined by

$$\begin{aligned} \mu^+(dt, du, d\omega) &= N(\omega)(dt, du) \mathbb{P}(d\omega), \\ \mu^-(dt, du, d\omega) &= \mathcal{L}(dt, du) \mathbb{P}(d\omega), \end{aligned}$$

where \mathcal{L} is the Lebesgue measure on the strip $[0, T] \times \mathbb{R}_+$ while $N(\omega)$ is the discrete measure on this strip given by the Poisson process N . Let \mathcal{I} be the σ -field on $[0, T] \times \mathbb{R}_+ \times \Omega$ generated by the class

$$\mathcal{J} = \{E \times A; E \in \mathcal{B}([0, T] \times \mathbb{R}_+), A \in \mathcal{F}_{E^c}\}$$

where E^c denotes the complement of E . We claim that μ^+ and μ^- coincide on \mathcal{I} . This is in fact the statement of [14, Theorem 1]; we recall the short

proof here for completeness. Since \mathcal{J} is a π -system (i.e. stable by finite intersections) and $\sigma(\mathcal{J}) = \mathcal{I}$, it is enough to prove that μ_+ and μ_- coincide on \mathcal{J} . On the other hand, if $E \times A \in \mathcal{J}$, the random variable $N(E)$ is independent of the set A , hence

$$\mu^+(E \times A) = \mathbb{E}[N(E)\mathbb{1}_A] = \mathbb{E}[N(E)]\mathbb{P}(A) = \mathcal{L}(E)\mathbb{P}(A) = \mu^-(E \times A).$$

Next recall the definition of the predictable σ -field \mathcal{P} and observe that

$$\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{P} \subseteq \mathcal{I}.$$

As a result, since (H_t) and (λ_t) are predictable, as a function of (t, u, ω) ,

$$H_t \mathbb{1}_{\{u \leq \lambda_t\}}$$

is measurable with respect to \mathcal{I} . We may therefore integrate $H_t \mathbb{1}_{\{u \leq \lambda_t\}}$ with respect to μ^+ or μ^- and obtain the same outcome. In other words,

$$(40) \quad \mathbb{E} \left[\int_{[0, T] \times \mathbb{R}_+} H_t \mathbb{1}_{\{u \leq \lambda_t\}} N(dt, du) \right] = \mathbb{E} \left[\int_{[0, T] \times \mathbb{R}_+} H_t \mathbb{1}_{\{u \leq \lambda_t\}} dt du \right].$$

From (40) we obtain

$$\begin{aligned} \mathbb{E} \left[\int_0^T H_t X^\lambda(dt) \right] &= \mathbb{E} \left[\int_{[0, T] \times \mathbb{R}_+} H_t \mathbb{1}_{\{u \leq \lambda_t\}} N(dt, du) \right] \\ &= \mathbb{E} \left[\int_{[0, T] \times \mathbb{R}_+} H_t \mathbb{1}_{\{u \leq \lambda_t\}} dt du \right] = \mathbb{E} \left[\int_0^T H_t \lambda_t dt \right], \end{aligned}$$

completing the proof of Lemma 4.1.

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