Cantor's diagonalization method:
Proof of Shorack's Theorem 12.8.1

Jon A. Wellner

Let $I_n(t) \equiv \tau_{n,\lfloor nt \rfloor}/n$. For each fixed t we have

$$I_n(t) \to_p t \quad \text{by the weak law of large numbers.} \tag{1}$$

We want to show that

$$\|I_n - I\| \equiv \sup_{0 \le t \le 1} |I_n(t) - t| \to_p 0. \tag{2}$$

Shorack writes: "Using the diagonalization technique, we can extract from any subsequence a further subsequence $\{n'\}$ on which

$$I_{n'}(t) \to_{a.s.} t \quad \text{for all rational } t. \tag{3}$$

But since all functions involved are monotone, and since the limit function is continuous, this implies that a.s.

$$I_{n'}(t) \to t \ \text{ uniformly on } [0,1]. \tag{4}$$

Thus (2) follows from (4) since every (subsequence) n has a further (subsequence) n' with the same limit."

The key part of this argument is going from (1) to (3). Here are the details of that argument.

Let $r_1, r_2, \ldots$ be the rationals in $[0,1]$. Then $I_n(r_1) \to_p r_1$ by (1). By Theorem 3.5.7 (PfS 2017, page 61) there is a subsequence $\{n'\} \equiv \{n'_1\} \equiv \{n'_{1,j} : j \in \{1,2,\ldots\}\}$ such that $I_{n'_{1,j}}(r_1) \to_{a.s.} r_1$. A further subsequence $\{n'_2\}$ of $\{n'_1\}$ satisfies $I_{n'_{2,j}}(r_2) \to_{a.s.} r_2$. Continuing in this way yields a subsequence $\{n'_k\} = \{n'_{k,j} : \ j \in \{1,2,\ldots\}\}$ of $\{n'_{k-1}\}$ which satisfies $I_{n'_{k,j}} \to_{a.s.} r_k$, and so forth. Taking the diagonal subsequence $\{n'_{k,k} : k \in \{1,2,\ldots\}\}$ yields (3).

For another use of this method, see PfS, Theorem 9.1.3 (Helly's selection theorem), page 194.