Math/Stat 523, Spring 2020

Jon A. Wellner

Lecture 7

Monday April 20

Outline

- 1: Basic Limit Theorems: Gaussian and & Poisson Limits
- 2. Variations on the classical CLT
- 3: The "only if" part of the Lindeberg - Feller CLT

1. Basic Limit Theorems: Gaussian and Poisson limits

In the chapter 11 notes we used Lindeberg's successive replacement argument to prove several classical theorems with Gaussian limits. Here we study some of the same limit theorems using characteristic functions. We will also treat at least one interesting case with Poisson limits.

Theorem 1. (Classical CLT). Suppose that $X_{n1}, \ldots, X_{nn}$ are i.i.d. with $E(X_{n1}) = \mu$ and variance σ^2. Let $S_n = \sum_{j=1}^n X_{nj}$ and let $\overline{X}_n \equiv S_n/n$. Then

$$
\begin{aligned}
\sqrt{n}(\overline{X}_n - \mu) &= (S_n - n\mu)/\sqrt{n} = n^{-1/2} \sum_{j=1}^{n} (X_{nj} - \mu) \\
&\to_d \ \sigma Z \sim N(0, \sigma^2)
\end{aligned}
$$

where $Z \sim N(0,1)$.

Proof: The characteristic function of $\sqrt{n}(\overline{X}_n - \mu)$ is given by

$$\begin{aligned}
\phi_{\sqrt{n}(\overline{X}_n-\mu)}(t) &= \prod_{k=1}^{n} \phi_{(X_{nk}-\mu)/\sqrt{n}}(t) \\
&= [\phi_{(X_{n1}-\mu)/\sqrt{n}}(t)]^n \\
&= \Big[1 - \frac{\sigma^2}{2}\left(\frac{t}{\sqrt{n}}\right)^2 + \left(\frac{t}{\sqrt{n}}\right)^2 g(t/\sqrt{n})\Big]^n
\end{aligned}$$

by Inequality 9.6.2 with $m = 2$ and $g(t) \to 0$ as $t \to 0$. Let $\theta_{nk} \equiv -\frac{\sigma^2 t^2}{2n} + \frac{t^2}{n} g(t/\sqrt{n})$. Then, by the first product lemma

$$\begin{aligned}
\sum_{k=1}^{n} \theta_{nk} &= -\frac{\sigma^2 t^2}{2} + t^2 g(t/\sqrt{n}) \to -\frac{\sigma^2 t^2}{2}, \\
\max_{1\le k\le n} |\theta_{nk}| &\le \frac{\sigma^2 t^2}{2n} + (t^2/n) g(t\sqrt{n}) \to 0, \\
\sum_{k=1}^{n} |\theta_{nk}| &\le \frac{\sigma^2 t^2}{2} + t^2 g(t/\sqrt{n}) \le \text{ some } M.
\end{aligned}$$

Thus

$$\left[1 - \frac{\sigma^2}{2} \cdot \frac{t^2}{n} + \frac{t^2}{n} g(t/\sqrt{n})\right]^n \to \text{exp}(-\sigma^2 t^2/2).$$

That is, with $Z \sim N(0,1)$,

$$\phi_{\sqrt{n}(\overline{X}_n - \mu)}(t) \to e^{-\sigma^2 t^2/2} = \phi_{\sigma Z}(t).$$

Thus $\sqrt{n}(\overline{X}_n - \mu) \to_d \sigma Z$ by the Cramér - Lévy continuity theorem. Furthermore, by the second product lemma,

$$
\begin{aligned}
&\left|\phi_{\sqrt{n}(\overline{X}_n-\mu)}(t) - \left(1-\frac{\sigma^2 t^2}{2n}\right)^n\right| \\
&\quad = \left|\prod_{k=1}^n \phi_{(X_{nk}-\mu)/\sqrt{n}}(t) - \prod_{k=1}^n \left(1-\frac{\sigma^2 t^2}{2n}\right)\right| \\
&\quad \le \sum_{k=1}^n \left|\phi_{(X_{nk}-\mu)/\sqrt{n}}(t) - \left(1-\frac{\sigma^2 t^2}{2n}\right)\right| \\
&\quad \le \sum_{k=1}^n \frac{t^2}{n} g(t/\sqrt{n}) = t^2 g(t/\sqrt{n}) \to 0.
\end{aligned}
$$

But

$$
\left(1-\frac{\sigma^2 t^2}{2n}\right)^n \to e^{-\sigma^2 t^2/2} = \phi_{\sigma Z}(t),
$$

so the continuity theorem and uniqueness theorem complete the proof. $\square$

The following useful result is the multivariate version of the classical CLT.

Theorem (The multivariate CLT) Suppose that $\underline{X}_1, \ldots, \underline{X}_n$ are i.i.d. random vectors in $\mathbb{R}^d$ with mean $\underline{\mu}$ and covariance matrix $\Sigma = E[(\underline{X}_1 - \underline{\mu})(\underline{X}_1 - \underline{\mu})^T]$ and hence necessarily $E|\underline{X}_1|^2 < \infty$. Then

$$\sqrt{n}(\overline{\underline{X}}_n - \underline{\mu}) \rightarrow_d N_d(0, \Sigma). \tag{1}$$

Proof: For $\underline{\lambda} \in \mathbb{R}^d$, $Y_j \equiv \underline{\lambda}^T(\underline{X}_j - \underline{\mu})$ are i.i.d. $(0, \underline{\lambda}^T \Sigma \underline{\lambda})$ random variables. Thus by the classical CLT

$$\sqrt{n}\,\overline{Y}_n \;=\; \underline{\lambda}^T \sqrt{n}(\overline{\underline{X}}_n - \underline{\mu}) \rightarrow_d N_1(0, \underline{\lambda}^T \Sigma \underline{\lambda}) \stackrel{d}{=} \underline{\lambda}^T N_d(0, \Sigma).$$

Since this holds for all $\underline{\lambda} \in \mathbb{R}^d$, it follows from the Cramér-Wold device that (1) holds. $\square$

A Poisson Limit Theorem:

Now suppose that $X_{n,1}, \ldots, X_{n,n}$ are independent with $X_{n,k} \sim$ Bernoulli$(\lambda_{n,k})$. Let $\lambda_n \equiv \sum_{k=1}^n \lambda_{n,k}$, and let $S_n \equiv \sum_{k=1}^n X_{n,k}$. Furthermore, let $T_n \sim$ Poisson(λ_n), and for $A \subset \mathbb{N}$ write

$$P_n(A) \equiv P(S_n \in A) \qquad \text{and} \qquad Q_n(A) \equiv P(T_n \in A).$$

Then

$$d_{TV}(P_n, Q_n) \le \begin{cases} \sum_{k=1}^n \lambda_{n,k}^2 \le \max_{k \le n} \lambda_{n,k} \sum_{k=1}^n \lambda_{n,k} \\ 1.05 \frac{\sum_1^n \lambda_{n,k}^2}{\sum_1^n \lambda_{n,k}} \quad \text{if } \max_{k \le n} \lambda_{n,k} \le 1/4 \ . \end{cases}$$

Thus, if $\lambda_n \to \lambda > 0$ and $\sum_1^n \lambda_{n,k}^2 \to 0$, then

$$S_n \to \text{Poisson}(\lambda) \quad \text{and} \quad \max_{1 \le k \le n} |X_{n,k}| \to_d \text{Bernoulli}(1 - e^{-\lambda}).$$

Recall that

$$d_{TV}(P,Q) \equiv \sup_{A \in \mathcal{A}} |P(A) - Q(A)| = 2^{-1} \int |p - q| d\mu$$

if $P, Q \prec\prec \mu$ with densities $p = dP/d\mu$ and $q = dQ/d\mu$, and hence

$$d_{TV}(P_n, Q_n) = 2^{-1} \sum_{k=1}^{\infty} |P(S_n = k) - P(T_n = k)|.$$

Proof: (via Characteristic Functions).
Now

$$\phi_{n,k}(t) = 1 + \lambda_{n,k}(e^{it} - 1),$$

and hence

$$\phi_{S_n}(t) = \prod_{k=1}^{n} [1 + \lambda_{n,k}(e^{it} - 1)] \equiv \prod_{1}^{n} [1 + \theta_{n,k}]$$

where ...

$$
\begin{aligned}
\sum_1^n \theta_{n,k} &= \sum_1^n \lambda_{n,k}(e^{it}-1) = \lambda_n(e^{it}-1) \to \lambda(e^{it}-1), \\
\max_{k\le n} |\theta_{n,k}| &\le 2\max_{k\le n} \lambda_{n,k} \\
&= 2\left\{\max_{k\le n} \lambda_{n,k}^2\right\}^{1/2} \le 2\left\{\sum_1^n \lambda_{n,k}^2\right\}^{1/2} \\
&\to 0, \\
\sum_1^n |\theta_{n,k}| &\le 2\sum_1^n \lambda_{n,k} \le \text{ some } M.
\end{aligned}
$$

Thus by the product lemma,

$$\phi_{S_n}(t) \to \exp(\lambda(e^{it}-1)) \equiv \phi_T(t)$$

where $T \sim$ Poisson . Moreover,

$$
\begin{aligned}
P(\max_{k \le n} |X_{n,k}| \le t) &= P(X_{n,1} \le t, \ldots, X_{n,n} \le t) \\
&= \prod_{k=1}^{n} (1 - \lambda_{n,k}) \quad \text{for} \quad 0 \le t < 1 \\
&\to e^{-\lambda}.
\end{aligned}
$$

Since $\max_{k \le n} X_{n,k}$ takes on only the values 0 and 1, this yields $\max_{k \le n} \to_d$ Bernoulli$(1 - e^{-\lambda})$. $\square$

2. Variations on the classical CLT

Now let $X_{n,1},\ldots,X_{n,n}$, $n \geq 1$ be row independent rv's with means and variances $(\mu_{n,k},\sigma_{n,k}^2)$, $\gamma_{n,k} \equiv E|X_{n,k} = \mu_{n,k}|^3 < \infty$. Let

$$
\begin{aligned}
\sigma_n^2 &\equiv \sum_{k=1}^n \sigma_{n,k}^2, \qquad \gamma_n \equiv \sum_{k=1}^n \gamma_{n,k}, \\
Z_n &\equiv \sum_{k=1}^n (X_{n,k} - \mu_{n,k})/\sigma_n,
\end{aligned}
$$

so that $E(Z_n) = 0$, $Var(Z_n) = 1$. The distribution of Z_n may be complicated, though. Let

$$
\begin{aligned}
&\phi_{n,k}(t) \equiv E e^{it(X_{n,k}-\mu_{n,k})/\sigma_n} \\
&\phi_{Z_n}(t) = \prod_{k=1}^n \phi_{n,k}(t), \quad \text{and} \\
&F_{Z_n}(t) = P(Z_n \leq t).
\end{aligned}
$$

Theorem 10.2.1. (Rate of convergence in the CLT) With the notation introduced above

$$\|F_{Z_n} - \Phi\|_\infty \leq 13\frac{\gamma_n}{\sigma_n^3}.$$

(Recall that we proved $\|F_{Z_n} - \Phi\|_\infty \leq 9\gamma_n/\sigma_n^3$ in Chapter 11, Theorem 11.1.1.)

Corollary 1. (Liapunov's CLT) If $\gamma_n/\sigma_n^3 \to 0$, then $Z_n \to_d Z \sim N(0,1)$.

Corollary 2. (Berry - Esseen theorem). If $X_1, \ldots, X_n$ are i.i.d. with $\gamma \equiv E|X-\mu|^3 < \infty$, then

$$\|F_{Z_n} - \Phi\|_\infty \leq \frac{8\gamma/\sigma^3}{\sqrt{n}}.$$

Proof: The proof relies on Esseen's lemma.

First we write

$$
\begin{aligned}
\left|\phi_{Z_n}(t) - e^{-t^2/2}\right| &= \Big|\prod_{k=1}^{n} \phi_{n,k}(t) - e^{-t^2/2}\Big| \\
&\le \ e^{-t^2/2}\left|\mathrm{exp}\left\{\sum_{1}^{n} \mathrm{Log}\phi_{n,k}(t)\} + t^2/2\right\} \ - \ 1\right| \\
&\equiv \ e^{-t^2/2}\left|e^{z} - 1\right| \le e^{-t^2/2}|z|e^{|z|}
\end{aligned}
$$

for all z where

$$
\begin{aligned}
|z| &= \Big|\sum_1^n \mathrm{Log}\phi_{n,k}(t)\} + t^2/2\Big| \\
&= \Big|\sum_1^n \left\{\mathrm{Log}(1+[\phi_{n,k}(t)-1]) - \frac{i^2t^2\sigma_{n,k}^2}{2\sigma_n^2}\right\}\Big| \\
&\le \Big|\sum_1^n \left\{[\phi_{nk}(t) - \left(1+\frac{i^2t^2\sigma_{nk}^2}{2\sigma_n^2}\right)] \oplus |\phi_{nk}(t)-1|^2\right\}\Big| \\
&\qquad \text{provided } |\phi_{nk}(t)-1| \le 1/2 \\
&\le \frac{|t|^3}{6}\cdot\frac{\gamma_n}{\sigma_n^3} + \sum_{k=1}^n \left(K_{1,1/2}\frac{|t|^{3/2}E|X_{nk}|^{3/2}}{\sigma_n^{3/2}}\right)^2, \qquad (2) \\
&\qquad \text{by Inequality 9.4.1 with } m=2, \delta=1, \\
&\qquad \text{and again with } m=1,\ \delta=1/2
\end{aligned}
$$

where the second term in the last inequality of the previous display holds since, for $|z| \le 1/2$,

$$
\begin{aligned}
&\Big|\mathrm{Log}(1 + [\phi_{nk}(t) - 1]) - (\phi_{nk}(t) - 1)\Big| \\
&\quad \le \ |\phi_{nk}(t) - 1|^2 \\
&\quad \le \ \frac{|t|^3}{2a}\left\{\frac{1}{3} + \frac{8 \cdot 2}{9}\right\} = \frac{|t|^3}{2a}\left\{\frac{3+16}{9}\right\} \\
&\quad \le \ \frac{|t|^4}{4} \ \text{ on } |t| \le (9/38)a. \qquad (3)
\end{aligned}
$$

Here $a \equiv \sigma_n^3/\gamma_n$ and we have used

$$
\begin{aligned}
K_{1,1/2} &= \frac{(1/2)2^{1/2}}{(3/2)(1/2)} = \frac{2 \cdot 2^{1/2}}{3}, \quad \text{so that} \\
K_{1,1/2}^2 &= \frac{4 \cdot 2}{9} = \frac{8}{9}.
\end{aligned}
$$

Furthermore, we see that $|z| \leq 1/2$ holds by our basic chf expansion with $m = 1$ and $\delta = 1$ yields

$$|\phi_{n,k}(t) - 1| \leq K_{1,1}|t|^2 E|Z_{nk}|^2 = \frac{1}{2 \cdot 1}|t|^2 \frac{\sigma_{nk}^2}{\sigma_n^2} \leq 1/2 \tag{4}$$

since for any fixed t this holds for all $1 \leq k \leq n$ for n large using

$$\max_{k \leq n} \left(\frac{\sigma_{nk}^2}{\sigma_n^2} \right)^{3/2} \leq \frac{\gamma_{nk}}{\sigma_n^3} \leq \frac{\gamma_n}{\sigma_n^3} \to 0.$$

Thus for each fixed t

$$\phi_{Z_n}(t) \to \exp(-t^2/2) \quad \text{and} \quad Z_n \to_d Z \sim N(0,1).$$

whenever $\gamma_n/\sigma_n^3 \to 0$. This completes the proof of Corollary 1.

Returning to the proof of Theorem 10.2.1, consider (4) again.

If $|t| \le a^{1/3}$, then we see that

$$|\phi_{nk}(t) - 1| \le \frac{t^2}{2}\frac{\sigma_{nk}^2}{\sigma_n^2} \le \frac{1}{2}a^{2/3}\left(\frac{\gamma_n}{\sigma_n^3}\right)^{2/3} \le 1/2$$

on $0 \le |t| \le a^{1/3}$. Putting (3) and (4) together we have

$$|\phi_{Z_n}(t) - e^{-t^2/2}| \le \frac{19}{18a}|t|^3 e^{-t^2/4} \le \frac{2}{a}|t|^3 e^{-t^2/4}$$

for $|t| \le a^{1/3}$ when $a \ge 9$. But we need to extend this to $a^{1/3} \le t \le (3/8)a$.

To make this extension, note that $|\phi_n(t)|^2$ is the chf of the symmetrized rv $Z_n^s \equiv Z_n - Z_n'$ where Z_n, Z_n' are independent with the same distribution. This rv has mean 0, variance 2, and third absolute moment bounded above by $8\gamma_n/\sigma_n^3$ ...

(by the C_r-inequality with $r = 3$). Thus

$$
\begin{aligned}
|\phi_{Z_n}(t)| \le \left[|\phi_n(t)|^2\right]^{1/2} &\le \left[1 + 0 - \frac{2t^2}{2} + \frac{|t|^3}{3!}\frac{8\gamma_n}{\sigma_n^3}\right]^{1/2} \\
&\le \ \exp\left\{-t^2\left(\frac{1}{2} - \frac{2|t|}{3}\frac{\gamma_n}{\sigma_n^3}\right)\right\} \quad \text{using } 1 - x \le e{-}x \\
&\le \ \exp(-t^2/4) \quad \text{for} \quad |t| \le (3/8)a.
\end{aligned}
$$

Combining the various pieces we have

$$
|\phi_{Z_n}(t) - e^{-t^2/2}| \le \frac{2}{a}|t|^3 \exp(-t^2/4) \quad \text{for} \quad 0 \le |t| \le (3/8)a.
$$

Key chf inequality: Combining all the pieces yields

$$
|\phi_{Z_n}(t) - e^{-t^2/2}| \le (2|t|^3(\gamma_n/\sigma_n^3))e^{-t^2/4} \quad \text{for} \quad 0 \le |t| \le (3/8)a \quad (5)
$$

Now we can apply Esseen's lemma: we conclude that

$$
\begin{aligned}
\|F_{Z_n} - \Phi\|_\infty &\le \int_{-(3/8)a}^{(3/8)a} \frac{1}{\pi|t|} \cdot \frac{2|t|^3}{a} \exp(-t^2/4)dt + \frac{24/\sqrt{2\pi}}{\pi a} \\
&\le \frac{1}{a}\left\{\frac{2}{\pi}\int_{-\infty}^{\infty} |t|^2 \exp(-t^2/4)dt + \frac{c}{(3/8)a}\right\} \\
&= (1/a)\left\{\frac{8}{\sqrt{\pi}} + \frac{8c}{3}\right\} \doteq 12.641 a^{-1} \le 13/a.
\end{aligned}
$$

Here $c = 24/(\pi\sqrt{2\pi}) \doteq 3.04769$. $\square$

In the i.i.d. case, use $K_{1,1} = 1/2$ and $\beta = E|X|^3/\sigma^3 = \gamma/\sigma^3 \ge 1$ in (2), and obtain

$$
\begin{aligned}
|z| &\le \frac{|t|^3\beta}{6\sqrt{n}} + n\left(\frac{t^2\sigma^2}{2n\sigma^2}\right)^2 \le \frac{|t|^3\beta}{6\sqrt{n}} + \frac{t^4\beta^2}{4n} \\
&\le \frac{5}{12}\frac{\beta}{\sqrt{n}}|t|^3 \le \frac{5}{12}|t|^3 \quad \text{for all} \quad |t| \le \sqrt{n}/\beta,
\end{aligned}
$$

with (e) necessarily valid. Then (5) can be replaced in the i.i.d. case by

$$|\phi_{Z_n}(t) - e^{-t^2/2}| \le \frac{5}{12}\frac{\gamma}{\sigma^3\sqrt{n}}|t|^3 e^{-t^2/12} \quad \text{on} \quad 0 \le |t| \le \sqrt{n}\sigma^3/\gamma;$$

this leads to $8\gamma/\sqrt{n}\sigma^3$ when the steps leading to (I) are repeated. This completes the proof of Corollary 2. □

This brings us to a very important theorem concerning row - independent triangular arrays. We write $Z_{nk} \equiv (X_{nk} - \mu_{nk})/\sigma_n$ for $k \in \{1, \dots, n\}$.

Theorem. (Lindeberg - Feller). The following are equivalent:
A. $Z_n \to_d Z \sim N(0, 1)$ and for every $\epsilon > 0$,

$$\max_{1 \le k \le n} P(|X_{nk} - \mu_{nk}|/\sigma_n > \epsilon) \to 0 \quad \text{for every } \epsilon > 0.$$

B. For every $\epsilon > 0$,

$$L_n(\epsilon) \equiv \sum_{k=1}^{n} E\left\{|Z_{nk}|^2 1_{[|Z_{nk}| > \epsilon]}\right\} \to 0. \tag{6}$$

Proof. (Sufficiency; Lindeberg). The moment expansion 9.6.1 gives

$$\phi_{nk}(t) \equiv 1 + \theta_{nk}(t) \equiv 1 - \frac{\sigma_{nk}^2 t^2}{2\sigma_n^2} + \beta_{nk}(t).$$

Moreover,

$$\begin{aligned}
\phi_{Z_n}(t) &= \prod_{k=1}^{n} \phi_{nk}(t) = \prod_{k=1}^{n} [1 + \theta_{nk}(t)] \\
&= \prod_{k=1}^{n} \left[1 - \frac{\sigma_{nk}^2}{\sigma_n^2} \frac{t^2}{2} + \beta_{nk}(t) \right].
\end{aligned}$$

and

$$\begin{aligned}
\theta_n(t) &= \sum_{k=1}^{n} \theta_{nk}(t) = -\frac{t^2}{2} \sum_{k=1}^{n} \frac{\sigma_{nk}^2}{\sigma_n^2} + \sum_{1}^{n} \beta_{nk}(t) \\
&= -\frac{t^2}{2} + \sum_{1}^{n} \beta_{nk}(t).
\end{aligned}$$

Now we have

$$
\begin{aligned}
\Big|\sum_1^n \beta_{nk}(t)\Big| &= \Big|\sum_1^n [\phi_{nk}(t) - 1 - 0 + \frac{\sigma_{nk}^2}{\sigma_n^2}\frac{t^2}{2}]\Big| \\
&\le \Big|E\left\{e^{itZ_{nk}} - \Big(1 + itZ_{nk} + 2^{-1}(itZ_{nk})^2\Big)\right\}\Big| \\
&\le \sum_1^n \left\{E\frac{1}{6}\Big[|tZ_{nk}|^3 1_{[|Z_{nk}|\le\epsilon]}\Big]\right. \\
&\quad \left. + E\left[\frac{|t|^2}{2}E\Big[|Z_{nk}|^2 1_{[|Z_{nk}|>\epsilon]}\Big]\right]\right\} \\
&\le \epsilon\frac{|t|^3}{6} + \frac{t^2}{2}\sum_1^n E\left\{|Z_{nk}|^2 1_{[|Z_{nk}|>\epsilon]}\right\} \\
&\to \frac{\epsilon|t|^3}{6} \text{ as } n \to \infty.
\end{aligned}
$$

Thus asymptotic normality holds. Note that $\max_{k\le n}|\theta_{nk}(t)| \to 0$ as required by the product lemma, since we can use Inequality 9.6.1 on the θ_{nk}'s to claim that $|\theta_{nk}(t)| \le (t^2/2)(\sigma_{nk}^2/\sigma_n^2)$ and then use (6) on the second term below to see that

$$
\begin{aligned}
\frac{\sigma_{nk}^2}{\sigma_n^2} &\le E\Big[|Z_{nk}|^2 1_{[|Z_{nk}|\le\epsilon]}\Big] + E\Big[|Z_{nk}|^2 1_{[|Z_{nk}|>\epsilon]}\Big] \\
&\le \epsilon^2 + \sum_{k=1}^n E\Big[|Z_{nk}|^2 1_{[|Z_{nk}|>\epsilon]}\Big] \\
&\le \epsilon^2 + o(1) \le \epsilon \quad \text{for } n \ge \text{some } N_\epsilon.
\end{aligned}
$$

Note that this also implies that the second part of A holds as well:

$$
\begin{aligned}
\max_{1\le k\le n} P(|X_{nk}-\mu_{nk}|/\sigma_n > \epsilon^{1/4}) &\le \epsilon^{-1/2} \max_{1\le k\le n} \frac{\sigma_{nk}^2}{\sigma_n^2} \\
&\le \epsilon^{1/2} \quad \text{for } n \ge \tilde{N}_\epsilon.
\end{aligned}
$$

This completes the proof of suficiency; that is, B implies A. $\square$

Proof: (Necessity in the Lindeberg-Feller CLT) Now suppose that

$$Z_n \to_d N(0,1) \quad \text{and} \quad \max_{k\le n} P(|X_{nk}-\mu_{nk}|/\sigma_n > \epsilon) \to 0$$

for every $\epsilon > 0$. Applying Inequality 9.6.2 implies via PfS Exercise 10.2.1 that the terms $z_{nk} \equiv \phi_{nk}(t) - 1$ converge uniformly to 0 on any finite interval, and hence

$$\begin{aligned}
\Big|\sum_1^n \text{Log}\phi_{nk}(t) - \sum_1^n [\phi_{nk}(t) - 1]\Big| &\le \sum_{k=1}^n |\phi_{nk}(t) - 1|^2 \\
&\le \ [\max_{1\le k\le n} |\phi_{nk}(t) - 1| \cdot (t^2/2)[\sum_{k=1}^n \sigma_{nk}^2/\sigma_n^2] \ \text{ by (9.6.5)} \\
&\le \ o(1)\cdot(t^2/2)\cdot 1 \to 0 \ \text{ using A via Exercise 10.2.1.}
\end{aligned}$$

We thus have (for any finite m)

$$\text{Log} \prod_{k=1}^n \phi_{nk}(t) = \sum_1^n [\phi_{nk}(t) - 1] + o(1) \quad \text{uniformly on any } \ |t| \le M.$$

But we also know that $\text{Log}\prod_{k=1}^{n}\phi_{nk}(t) \to -t^2/2$, since we have assumed asymptotic normality. [Recall that $a = b \oplus c$ means that $|a-b| \le c$.] Combining the last two facts shows that for every tiny $\epsilon > 0$ and every huge $M > 0$ we have

$$-t^2/2 = \text{Real}(-t^2/2) = \text{Real}\{\sum_{1}^{n}[\phi_{nk}(t) - 1]\} \oplus \epsilon \quad \text{for} \quad |t| \le M$$

for all large n; that is, for $n \ge$ (some $N_{\epsilon,M}$) we have

$$t^2/2 = \sum_{1}^{n} E\{1 - \cos(tY_{nk})\} \oplus \epsilon \quad \text{on } |t| \le M.$$

Define $Y_{nk} \equiv \sigma_n Z_{nk}$. We also define $I_{nk} \equiv 1\{|Y_{nk}| < \epsilon\sigma_n\}$ and $I_{nk}^c \equiv 1\{|Y_{nk}| \ge \epsilon\sigma_n\}$. Note that

$$0 \le 1 - \cos(ty/\sigma_n) \le (t^2 y^2/2\sigma_n^2). \tag{7}$$

Thus for all $|t| \le M$ we have, for all $n \ge N_{\epsilon,M}$,

$$
\begin{aligned}
(t^2/2) \sum_{k=1}^{n} E\{I_{nk}^c (Y_{nk}^2/\sigma_n^2)\} &= (t^2/2) \left(1 - \sum_{k=1}^{n} E\{I_{nk}(Y_{nk}/\sigma_n)^2\} \right) \\
&= (t^2/2) \;-\; \sum_{k=1}^{n} E\{I_{nk}(t^2 Y_{nk}^2 / 2\sigma_n^2)\} \\
&\le (t^2/2) - \sum_{k=1}^{n} E\{I_{nk}(1 - \cos(tY_{nk}/\sigma_n))\} \\
&= \sum_{k=1}^{n} E\{I_{nk}^c (1 - \cos(tY_{nk}/\sigma_n))\} \oplus \epsilon \\
&\le 2 \sum_{k=1}^{n} E\{I_{nk}^c\} + \epsilon \\
&= 2 \sum_{k=1}^{n} P(|Z_{nk}| \ge \epsilon) + \epsilon \quad \le \quad 2\epsilon^{-2} + \epsilon.
\end{aligned}
$$

On the other hand we also have

$$\begin{aligned} 2\sum_{k=1}^{n} E\{I_{n,k}^c\} \ + \ \epsilon &\le (2/\epsilon^2)\sum_{k=1}^{n} E\{I_{nk}^c Y_{nk}^2/\sigma_n^2\} + \epsilon \\ &\le (2/\epsilon^2) + \epsilon. \end{aligned}$$

Choosing $t^2 = M^2 = 4/(\theta\epsilon^2)$ for any fixed $0 < \theta < 1$ yields, for $n \ge N_{\epsilon,\theta}$,

$$\begin{aligned} L_n(\epsilon) &= \frac{1}{\sigma_n^2}\sum_{k=1}^{n} E\{Z_{nk}^2 1_{[|Z_{nk}|^2 > \epsilon]}\} \\ &= [(2/\epsilon)^2 + \epsilon] \cdot 2^{-1}\epsilon^2\theta = \theta(1 + \epsilon/2) \le 2\theta. \end{aligned}$$

Thus the Lindeberg condition B holds.

Remark: (i) If Lindeberg's condition fails, it may still be true that

$$S_n/\sigma_n \to_d N(0,a^2) \quad \text{with} \quad a^2 < 1 \quad \text{and} \quad \max_{k\le n} \frac{\sigma_{nk}^2}{\sigma_n^2} \to 0.$$

Let $Y_1, Y_2, \ldots$ be i.i.d. $(0,1)$ random variables so that $\sqrt{n}\overline{Y}_n \to_d N(0,1)$ by the CLT. Now let the rv's U_k be independent $(0,c^2)$ with U_k taking the values $-ck$, 0, and ck with probabilities $1/(2k^2)$, $1-1/k^2$, $1/(2k^2)$. Since $\sum_1^\infty P(|U_k| \ge \epsilon) = \sum_{k=1}^\infty k^{-2} < \infty$, the Borel-Cantelli lemma shows that for a.e. ω the sequence $\{U_k\}$ satisfies $U_k \ne 0$ only finitely often. Thus $\sqrt{k}U_k \to_{a.s.} 0$. For $n \ge 1$ set $X_n = Y_n + U_n$ and let $S_n \equiv X_1 + \cdots + X_n$. Note that $\sigma_n^2 = Var(S_n) = (1+c^2)n$, so by Slutsky's theorem

$$\begin{aligned} S_n/\sigma_n &= (\sqrt{n}\,\overline{Y}_n)/\sqrt{1+c^2} + (\sqrt{n}\,\overline{U}_n)/\sqrt{1+c^2} \\ &\to_d Z/\sqrt{1+c^2} \sim N(0, 1/(1+c^2)) \\ &= N(0,a^2) \quad \text{with} \quad a^2 = 1/(1+c^2) < 1. \end{aligned}$$

Note that $\max_{k\le n}\sigma_{nk}^2/\sigma_n^2 \to 0$. However, the Lindeberg condition fails:

$$
\begin{aligned}
L_n(\epsilon) &= \frac{a^2}{n}\sum_{k=1}^{n} E\{|X_{nk}|^2 1_{[|X_{nk}|>\epsilon\sqrt{n}/a]}\} \\
&\sim \frac{a^2}{n}\sum_{k:ck\ge\epsilon\sqrt{n}/a}\frac{(kc)^2}{k^2} + o(1) \\
&\sim \frac{c^2}{1+c^2}\frac{1}{n}\sum_{k:ck\ge\epsilon\sqrt{n}/a} 1 \to \frac{c^2}{1+c^2} > 0;
\end{aligned}
$$

the non-zero contribution shown in the last step is due to the U_k's, whereas we know already that their contribution to the limit distribution is $o(1)$.

(ii) Note that if $X_{n1}\sim N(0,pn)$ for some $0<p<1$, $X_{nk}\equiv 0$ for $2\le k\le [pn]$, and $X_{nk}\sim N(0,1)$ for $pn<k\le n$ for independent rv's X_{nk}, then $S_n/\sigma_n \to_d N(0,1)$ while Lindeberg's condition fails and $\max_{k\le n}\sigma_{nk}^2/\sigma_n^2 \to p$.