Math/Stat 523, Spring 2020

Jon A. Wellner

Lecture 6

Wednesday April (15?)

More Characteristic Function Tools

- 1: Taylor expansions and Moment Expansions
- 2: Alternative tools
- 3: Esseen's lemma
- 4: Distributions on grids

1. Taylor expansions and Moment Expansions

First some basic facts about $\log(1+z)$ and e^z.

Lemma. (Taylor expansions of $\log(1+z)$ and e^z). First, note that $\log z$ is a many–valued function of a complex $z = re^{i\theta}$: any of $(\log r + i(\theta + 2\pi m)$ for $m = 0, \pm 1, \pm 2, \ldots$ will work for $\log z$. But when we write $\log z = \log r + i\theta$ we will always suppose that $-\pi < \theta \leq \pi$ Furthermore we donte this unique determination by $Log(z)$; this is the *principal branch*. The Taylor series expansion of $Log(1+z)$ gives

$$\left|\text{Log}(1+z) - \sum_{k=1}^{m-1}(-1)^{k-1}\frac{z^k}{k}\right| = \left|\sum_{k=m}^{\infty}(-1)^{k-1}\frac{z^k}{k}\right| \\ \leq \frac{|z|^m}{m}\left(1 + |z| + |z|^2 + \cdots\right) \leq \frac{|z|^m}{m(1-|z|)}$$

for $|z| < 1$. Thus

$$|\text{Log}(1+z) - z| \leq |z|^2/(2(1-\theta)) \quad \text{for } |z| < \theta - 1. \tag{1}$$

From another Taylor expansion we have for all z that

$$\Big|e^z - \sum_{k=0}^{m-1} \frac{z^k}{k!}\Big| = \Big| \sum_{k=m}^{\infty} \frac{z^k}{k!}\Big| \le |z|^m \sum_{j=0}^{\infty} \frac{|z|^j}{j!} \frac{j!}{(j+m)!} \le \frac{|z|^m e^{|z|}}{m!}.$$

Lemma 9.6.2. (Taylor expansion of e^{it}) Let $m \ge 0$ and $0 \le \delta \le 1$ (and set the constant $K_{0,0} = 2$ below). Then for all real t we have

$$\begin{aligned} \left|e^{it} - \sum_{k=0}^{m} \frac{(it)^k}{k!}\right| &\le \frac{\delta 2^{1-\delta}}{(m+\delta)\cdots(2+\delta)(1+\delta)(0+\delta)} |t|^{m+\delta} \\ &\equiv K_{m,\delta}|t|^{m+\delta}. \qquad (2) \end{aligned}$$

Proof. By induction. For $m = 0$ we have both

$$|e^{it} - 1| \leq 2 \leq 2|t/2|^{\delta} \quad \text{for } |t/2| \geq 1,$$

and (since $\int_0^t ie^{is}ds = \int_0^t (i\cos(s) - \sin(s))ds = e^{it} - 1$)

$$|e^{it} - 1| \leq \Big| \int_0^t ie^{is}ds \Big| \leq \int_0^{|t|} ds = |t| \leq 2|t/2|^{\delta} \quad \text{for} \quad |t/2| \leq 1;$$

so (2) holds for $m = 0$. We now assume that (2) holds for $m - 1$, and we will verify that it thus holds for m. We again use $e^{it} - 1 = \int_0^t ie^{is}ds$ and note that

$$i \sum_{k=0}^{m-1} \int_0^t [(is)^k/k!]ds = \sum_{k=0}^{m-1} (i^{k+1}/k!) \int_0^t s^k ds = \sum_{1}^{m} (it)^k/k!$$

to obtain

$$\begin{aligned}\left|e^{it} - \sum_{k=0}^{m} (it)^k/k!\right| &= \left|i\int_0^t \left\{e^{is} - \sum_{k=0}^{m-1}(is)^k/k!\right\} ds\right| \\ &\leq K_{m-1,\delta}\int_0^{|t|} s^{m-1+\delta} ds \quad \text{by the induction step} \\ &\leq K_{m,\delta}|t|^{m+\delta}.\end{aligned}$$

This completes the proof of (2). □

Inequality 9.6.1. (Moment expansion inequality) Suppose $E|X|^{m+\delta} < \infty$ for some $m \geq 0$ and $0 \leq \delta \leq 1$. Then

$$\left|\phi(t) - \sum_{k=0}^{m} \frac{(it)^k}{k!} EX^k\right| \leq K_{m,\delta}|t|^{m+\delta} E|X|^{m+\delta} \quad \text{for all } t.$$

This follows immediately from Lemma 9.6.2 by replacing t by tX and taking expectations.

2. Further tools

Lemma 9.6.3. (The first product lemma) For all $n \geq 1$ suppose that the complex numbers $\beta_{n1}, \ldots, \beta_{nn}$ satisfy

$$
\begin{aligned}
&(a) \quad \beta_n \equiv \sum_{k=1}^{n} \beta_{nk} \to \beta \quad \text{as} \quad n \to \infty. \\
&(b) \quad \delta_n \equiv \max_{1 \leq k \leq n} |\beta_{nk}| \to 0. \\
&(c) \quad M_n \equiv \sum_{k=1}^{n} |\beta_{nk}| \quad \text{satisfies} \quad \delta_n M_n \to 0.
\end{aligned}
$$

Then

$$\prod_{k=1}^{n} (1 + \beta_{nk}) \to e^{\beta} \quad \text{as} \quad n \to \infty. \tag{3}$$

[This should be compared with Lemma 8.1.4.]

Proof: When $0 < \delta_n \le 1/2$ (and we are on the principal branch) (1) gives

$$\Big| \sum_{k=1}^{n} \mathrm{Log}(1+\beta_{nk}) - \sum_{k=1}^{n} \beta_{nk} \Big| \le \sum_{k=1}^{n} |\beta_{nk}|^2 \le \delta_n M_n \to 0$$

by (c). Thus

$$\sum_{k=1}^{n} \mathrm{Log}(1+\beta_{nk}) \to \beta \quad \text{as} \quad n \to \infty.$$

The last display shows that

$$\begin{aligned} \prod_{k=1}^{n} (1+\beta_{n,k}) &= \exp(\mathrm{Log} \prod_{k=1}^{n} (1+\beta_{n,k})) \\ &= \exp\left(\sum_{k=1}^{n} \mathrm{Log}(1+\beta_{nk}) \right) \to \exp(\beta), \end{aligned}$$

and hence (3) holds. □

Lemma 9.6.4 (The second product lemma) If $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ denote complex numbers with modulus at most 1, then

$$\left| \prod_{j=1}^{n} z_k - \prod_{j=1}^{n} w_k \right| \leq \sum_{k=1}^{n} |z_k - w_k|.$$

Proof: This is trivial for $n = 1$. We will prove that it holds for $n > 1$ by induction. Now

$$\begin{aligned} | \prod_{k=1}^{n} z_k - \prod_{k=1}^{n} w_k | &\leq |z_n| \left| \prod_{k=1}^{n-1} z_k - \prod_{k=1}^{n-1} w_k \right| + |z_n - w_n| \left| \prod_{k=1}^{n-1} w_k \right| \\ &\leq \left| \prod_{k=1}^{n-1} z_k - \prod_{k=1}^{n-1} w_k \right| + |z_n - w_n| \prod_{k=1}^{n-1} 1 \\ &\leq \sum_{k=1}^{n-1} |z_k - w_k| + |z_n - w_n| \end{aligned}$$

by the induction step. □

Inequality 9.6.2 (Moment expansions of chfs) Suppose $0 < E|X|^m < \infty$ for some $m \geq 0$. Then, for some function g with $0 \leq g(t) \leq 1$, the chf ϕ of X satisfies,

$$\left|\phi(t) - \sum_{k=0}^{m} \frac{(it)^k}{k!} E(X^k)\right| \leq \frac{3}{m!} |t|^m E|X|^m g(t) \tag{4}$$

where $g(t) \to 0$ as $t \to 0$.

Proof: Use the real expansions for sin and cos to obtain

$$\begin{aligned} e^{itx} &= \cos(tx) + i\sin(tx) \\ &= \sum_{k=0}^{m-1} \frac{(itx)^k}{k!} + \frac{(itx)^m}{m!}\{\cos(\theta_1 tx) + i\sin(\theta_2 tx)\} \\ &= \sum_{k=0}^{m} \frac{(itx)^k}{k!} + \frac{(itx)^m}{m!}\{\cos(\theta_1 tx) + i\sin(\theta_2 tx) - 1\}. \end{aligned}$$

Here we have $0 \leq |\theta_1| \vee |\theta_2| \leq 1$.

Then (4) follows from the second line in the last display since

$$\lim_{t\to 0} E|X^m\{\cos(\theta_1 tX) - 1 + i\sin(\theta_2 tX)\}|$$
$$\leq \lim_{t\to 0} E\{|X|^m|\cos(\theta_1 tX) - 1 + i\sin(\theta_2 tX)|\} = 0$$

by the DCT with dominating function $3|X|^m$. □

Results from Fourier Analysis

Lemma 9.6.5 (Riemann - Lebesgue lemma).
If $\int_{-\infty}^{\infty}|g(x)|dx < \infty$, then

$$\int_{-\infty}^{\infty} e^{itx}g(x)dx \to 0 \quad \text{as} \quad t\to\infty.$$

Proof: Let λ denote Lebesgue measure on $\mathbb{R}$. The class of functions

$$\Psi \equiv \left\{ \psi \equiv \sum_1^m c_i 1_{(a_i,b_i]} : \ a_i, b_i \in \mathbb{R}, \ m \geq 1 \right\}$$

is dense in $L_1(\lambda)$ by Theorem 3.5.8. That is, if $\int_{-\infty}^{\infty} |g(x)|dx < \infty$ and $\epsilon > 0$, then there exists $\psi \in \Psi$ such that $\int_{-\infty}^{\infty} |g - \psi|dx < \epsilon$. Thus $\gamma(t) \equiv |\int_{-\infty}^{\infty} e^{itx} g(x)dx|$ satisfies

$$\begin{aligned} \gamma(t) &\leq \int_{-\infty}^{\infty} |e^{itx}||g(x) - \psi(x)|dx + \left| \int_{-\infty}^{\infty} e^{itx}\psi(x)dx \right| \\ &\leq \epsilon + \sum_1^m |c_i| \cdot \left| \int_{a_i}^{b_i} e^{itx} dx \right|. \end{aligned}$$

It thus suffices to show that for any $a, b \in \mathbb{R}$ we have

$$\int_a^b e^{itx} dx \to 0 \quad \text{as} \quad |t| \to \infty$$

But by writing $e^{itx} = \cos(tx) + i \sin(tx)$ and noting that

$$\int_a^b \cos(tx)dx = \frac{1}{t}\int_{at}^{bt} \cos(v)dv = t^{-1}\{\sin(bt) - \sin(at)\} \to 0$$

and

$$\int_a^b \sin(tx)dx = \frac{1}{t}\int_{at}^{bt} \sin(v)dv = t^{-1}\{-\cos(bt) + \cos(at)\} \to 0$$

as $|t| \to \infty$. □

Lemma 6.6: (Tail behavior of chf's).

(i) If F has density f with respect to Lebesgue measure, then

$$|\phi(t)| \to 0 \quad \text{as} \quad |t| \to \infty.$$

(ii) If F has $n+1$ integrable derivatives $f, f', \ldots, f^{(n)}$ on $\mathbb{R}$, then

$$|t|^n |\phi(t)| \to 0 \quad \text{as} \quad |t| \to \infty.$$

Proof: That $|\phi(t)| \to 0$ as $|t| \to \infty$ follows from the Riemann-Lebesgue lemma since f is integrable. Since f is absolutely continuous and is a density, it follows that $f(x) \to 0$ as $|x| \to \infty$. Then note that

$$\begin{aligned}
\phi(t) &= \int e^{itx} f(x)dx = \int f(x)d(e^{itx}/it) \\
&= (e^{itx}/it)f(x)|_{-\infty}^{\infty} - \int e^{itx} f'(x)dx/(it) \\
&= -\int e^{itx} f'(x)dx/(it) \quad \text{with} \quad f'(\cdot) \in L_1(\lambda)
\end{aligned}$$

using $f(x) \to 0$ as $|x| \to \infty$ in going from the second to third line above. Applying the Riemann-Lebesgue lemma to the last line yields $|t||\phi(t)| \to 0$ as $|t| \to \infty$. To complete the proof, continue integrating by parts and applying the Riemann-Lebesgue lemma.
$\square$

3: Esseen's lemma

Suppose that:

- G is a fixed function on $\mathbb{R}$ satisfying:
  - ▷ $G(-\infty) = 0$, $G(\infty) = 1$.
  - ▷ $G' \equiv g$ exists with $|g(\cdot)| \le M$.
  - ▷ $\int_{\mathbb{R}} x g(x) dx = 0$
- Let $\psi(t) \equiv \int_{\mathbb{R}} e^{itx} g(x) dx$.

Let F denote a general d.f. having mean 0, and let ϕ denote the characteristic function of g. Our goal is to bound $\|F - G\|_\infty \equiv \sup_{-\infty<x<\infty} |F(x) - G(x)|$ in terms of the distance between ϕ and ψ.

Inequality 9.7.1. (Esseen's lemma). Let F and G be as above. For any $a > 0$ the following uniform bound holds:

$$\|F - G\|_\infty \le \frac{1}{\pi} \int_{-a}^{a} \left| \frac{\phi(t) - \psi(t)}{t} \right| dt + \frac{24\|g\|}{\pi a}.$$

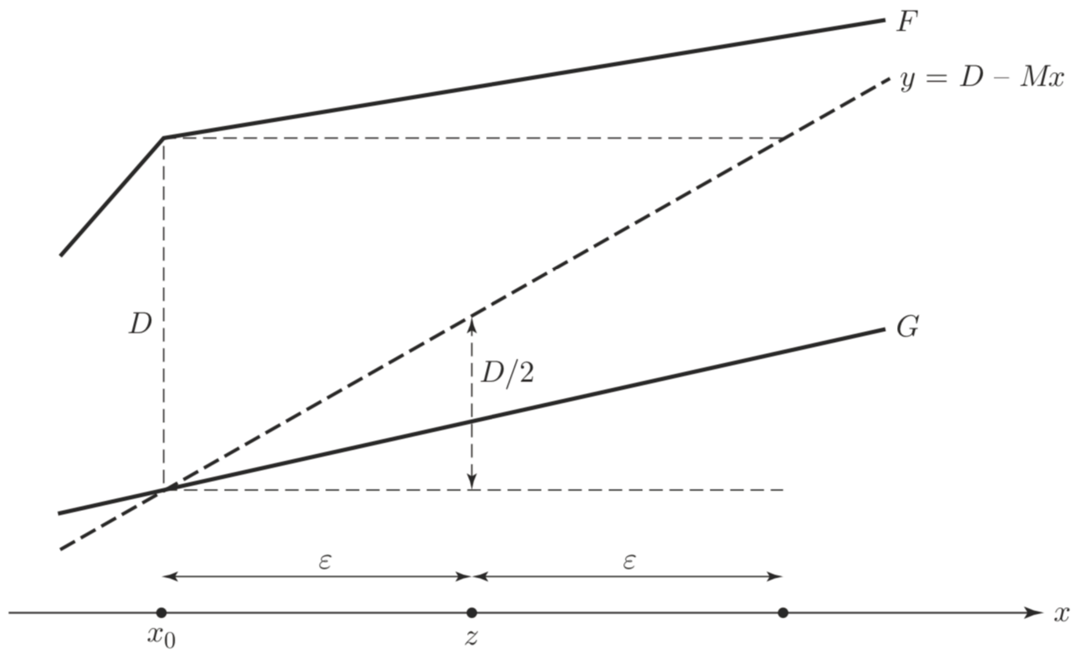
Proof: The key to the method is to smooth by convolving F and G with the d.f. H_a with density h_a and characteristic function γ_a given by

$$h_a(x) = \frac{1 - \cos(ax)}{\pi a x^2} \quad \text{on } \mathbb{R} \quad \text{and} \quad \gamma_a(t) = (1 - |t|/a) 1_{[-a,a]}(t).$$

The density h_a is that of V/a where V has the de la Vallée - Poussin density. Let F_a and G_a denote the convolutions of F and G with H_a for a large. We will now show that

$$\|F - G\|_\infty \leq 2\|F_a - G_a\|_\infty + 24\|g\|_\infty/(\pi a).$$

Let $\Delta \equiv F - G$. Then $\Delta(x) = \Delta_+(x)$and $\Delta_-(x)$ exist for all x. Thus there exists x_0 such that either $D \equiv \|F - G\|_\infty = |\Delta(x_0)|$ or $D = |\Delta_-(x_0)|$. without loss of generality, we suppose that $D = |\Delta(x_0)|$; if not replace X, Y by $-X, -Y$. Note Figure 7.1.



PfS, Figure 7.1: Bounds for Esseen's lemma

Without loss of generality, we act below as though $\Delta(x_0) > 0$, and we let $z_0 > x_0$. (If $\Delta(x_0) < 0$, then let $z_0 < x_0$. Since F is $\nearrow$ and g is bounded by M, it follows that for $|x| \le \epsilon$

$$
\begin{aligned}
\Delta(z_0 - x) &= F(z_0 - x) - G(z_0 - x) \\
&\ge F(z_0 - \epsilon) - G(x_0) - (G(z_0 - x) - G(x_0)), \\
&\qquad \text{since } z_0 - x \ge z_0 - \epsilon \\
&\ge F(x_0) - G(x_0) - \int_{x_0}^{z_0 - x} g(y)dy \quad \text{since } z_0 - \epsilon = x_0 \\
&\ge D - (z_0 - x - x_0)\|g\|_\infty \quad \text{since } \Delta(x_0) = D, \quad \|g\|_\infty \equiv M, \\
&= D/2 + xM \quad \text{since } z_0 - x_0 = \epsilon = D/(2M).
\end{aligned}
$$

Since D is the supremum,

$$\Delta(z_0 - x) \ge -D \quad \text{for} \quad |x| > \epsilon.$$

Thus with $\Delta_a \equiv F_a - G_a$, using the inequalities in the last two displays we find that

$$\begin{aligned}
\|F_a - G_a\| &\geq \Delta_a(z_0) = \int_{-\infty}^{\infty} \Delta(z_0 - x) h_a(x)dx \\
&\qquad\qquad \text{by the convolution formula} \\
&\geq \int_{-\epsilon}^{\epsilon} [D/2 + Mx] h_a(x)dx - D \int_{[|x|>\epsilon]} h_a(x)dx \\
&= (D/2)[1 - \int_{[|x|>\epsilon]} h_a(x)dx] + M \cdot 0 - D \int_{[|x|>\epsilon]} h_a(x)dx \\
&\qquad\qquad \text{since} \ \ xh_a(x) \ \ \text{is odd} \\
&= (D/2) - (3D/2) \int_{[|x|>\epsilon]} h_a(x)dx \geq (D/2) - 12M/(\pi a) \\
&= \|F - G\|_\infty / 2 - (12M/\pi a),
\end{aligned}$$

since

$$\int_{[|x|>\epsilon]} h_a(x)dx \leq 2 \int_x^{\infty} (2/\pi a x^2)dx = 4/(\pi a \epsilon) = 8M/(\pi a D).$$

We now bound $\|F_a - G_a\|_\infty$, By the Fourier inversion formula, F_a and G_a have bounded continuous "densities" that satisfy

$$f_a(x) - g_a(x) = \frac{1}{2\pi} \int_{-a}^{a} e^{-itx} [\phi(t) - \psi(t)|] \gamma_a(t) dt. \tag{5}$$

Formal integration of this identity over $(-\infty, x]$ leads us to conjecture that

$$\Delta_a(x) = \frac{1}{2\pi} \int_{-a}^{a} e^{-itx} \frac{\phi(t) - \psi(t)}{-it} \gamma_a(t) dt. \tag{6}$$

That the integrand is a continuous function that equals 0 at $t = 0$ (this follows from Inequality 9.6.1 since F and G have 0 "means") makes the right side well-defined and we may differentiate under the integral sign by the DCT (with dominating function $\gamma_a(t)$) to recover the identity (5). Thus $\Delta_a(x)$ can differ from the right side by at most a constant.

But this constant is 0 since $\Delta_a(x) \to 0$ as $|x| \to \infty$ while the same is true for the right side by the Riemann-Lebesgue lemma. But now (6) yields

$$|\Delta_a(x)| \leq \frac{1}{2\pi} \int_{-a}^{a} \left| \frac{\phi(t) - \psi(t)}{t} \right| dt \quad \text{for all} \quad x.$$