Math/Stat 523, Spring 2020

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Lecture 4

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Outline

- 1: Variation and (non-)differentiability; the Hölder condition
- 2: Structure of the zero set.
- 3: Strassen's functional LIL
- 4: Boundary crossing probabilities

1. Variation and (non-)differentiability

Definition 12.9.1: (Variation) For a sequence of partitions

$$\mathcal{P}_n \equiv \{(t_{n,k-1}, t_{n,k}] \ : \ k = 1, \ldots, n\} \quad \text{of} \quad [0,1]$$

with $0 \leq t_{n,0} < \cdots < t_{nn} \equiv 1$, define the rth *variation* of $\mathbb{S}$ corresponding to $\mathcal{P}_n$ by

$$V_n(r) \equiv \sum_{k=1}^{n} |\mathbb{S}(t_{n,k}) - \mathbb{S}(t_{n,k-1})|^r.$$

We call these partitions *nested* if $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \geq 1$. We also define the *mesh* of the partition to be $\|\mathcal{P}_n\| \equiv \sup_{1 \leq k \leq n} |t_{nk} - t_{n,k-1}|$.

2. Non-differentiability, variation, and Hölder condition

Theorem 12.9.1: (Non-differentiability)
(a) Almost every Brownian path is nowhere differentiable.
(b) In fact, $V_n(1) \to_{a.s.} \infty$ a.s. if $\|\mathcal{P}_n\| \to 0$.
(c) (Finite squared variation) $V_n(2) \to_2 1$ if $\|\mathcal{P}_n\| \to 0$.
(d) (Finite squared variation) $V_n(2) \to_{a.s.} 1$ if either
$(i) \sum_n \|\mathcal{P}_n\| < \infty$ or (ii) the $\mathcal{P}_n$ are nested with $\|\mathcal{P}_n\| \to 0$.
(e) (Dudley) $V_n(2) \to_{a.s.} 1$ if and only if $(\log n)\|\mathcal{P}_n \to 0$.

Partial proof of Theorem 12.9.1:

(a) If $t \mapsto \mathbb{S}(t, \omega)$ is differentiable at some point $s \in [0, 1]$, then

$$A_n \equiv \{|\mathbb{S}(t) - \mathbb{S}(s)| < C(t - s), \quad s < t < s + 5/n\}, \quad \text{for } n \geq m$$

for some $1 \leq C < \infty$ and some $m \geq 1$. But this is a subset of the event

$$B_n \equiv \bigcup_{0<i\leq n+2} \bigcap_{i<k\leq i+3} \left\{ \left| \mathbb{S}\left(\frac{k}{n}\right) - \mathbb{S}\left(\frac{k-1}{n}\right) \right| < \frac{7C}{n} \right\}.$$

Then, since

$$\mathbb{S}\left(\frac{k}{n}\right) - \mathbb{S}\left(\frac{k-1}{n}\right) \overset{d}{=} n^{-1/2} Z \sim N(0, 1/n),$$

it follows that

$$\begin{aligned} P(A_n) \leq P(B_n) &\leq (n+2) P\left(\left| \mathbb{S}\left(\frac{1}{n}\right) \right| < \frac{7C}{n} \right)^3 \\ &= (n+2) P\left(|Z| < 7C/\sqrt{n} \right)^3 \\ &\leq M n^{-1/2} \to 0 \end{aligned}$$

for some constant M (and $M = 14C/\sqrt{2\pi}$ works).

(c) For $r = 2$, note that

$$\begin{aligned} EV_n(2) &= E\{\sum_{k=1}^n |\mathbb{S}(t_{n,k}) - \mathbb{S}(t_{n,k-1})|^2\} \\ &= \sum_{k=1}^n E|\mathbb{S}(t_{n,k}) - \mathbb{S}(t_{n,k-1})|^2 \\ &= \sum_{k=1}^n (t_{n,k} - t_{n,k-1}) = 1. \end{aligned}$$

Thus $E|V_n(2) - 1|^2 = Var(V_n(2)) \le (C_4 - C_2^2)\|\mathcal{P}_n\| \to 0$ if $\|\mathcal{P}_n\| \to 0$. Here the variance calculation follows easily using $E|Z|^r \equiv C_r = 2^{r/2}\Gamma((r+1)/2)/\sqrt{\pi}$ for $Z \sim N(0,1)$ and $r > 0$. Hence

$$\begin{aligned} E|\mathbb{S}(t_{n,k-1}, t_{n,k}]|^r &= C_r(t_{n,k} - t_{n,k-1})^{r/2}, \\ Var[|\mathbb{S}(t_{n,k-1}, t_{n,k}]|^r] &= (C_{2r} - C_r^2)(t_{n,k} - t_{n,k-1})^r. \end{aligned}$$

Theorem 12.9.2: (Lévy) The following Hölder type condition holds:

$$\limsup_{0 \le s < t \le 1:\ t-s=a \searrow 0} \frac{|\mathbb{S}(t) - \mathbb{S}(s)|}{\sqrt{2a\log(1/a)}} = 1 \quad \text{a.s.}$$

2. Structure of the zero set.

Theorem 12.9.3: (The zeros of $\mathbb{S}$ in $[0,1]$) Define

$$Z(\omega) = \{t \in [0,1] : \ \mathbb{S}(t,\omega) = 0\}.$$

For almost all ω the set $Z(\omega)$ is a closed and perfect set of Lebesgue measure zero. [A set A is called *dense in itself* if every point x of A is such that every neighborhood of x contains another point of A; i.e. no point of A is isolated. A closed set dense in itself is called *perfect*.]

Partial proof: Note that $\lambda(Z(\omega)) = \int_0^1 1_0(\mathbb{S}(t,\omega))dt$, so

$$E\lambda(Z) = E\int_0^1 1_0(\mathbb{S}(t))dt = \int_0^1 P(\mathbb{S}(t) = 0)dt = \int_0^1 0 \cdot dt = 0$$

by the Tonelli part of Fubini's theorem. Thus $\lambda(Z(\omega)) = 0$ a.s.

3. Strassen's functional LIL .

Strassen's functional LIL is to the ordinary LIL as Donsker's (functional central limit) theorem is to the ordinary CLT.

Notation:

- Let $\mathbb{S}$ be standard Brownian motion.
- Let $\mathbb{Z}_n(t) \equiv \mathbb{S}(nt)/\sqrt{2n\text{loglog}n}$ for $0 \le t \le 1$.
- Let $\mathcal{K}$ denote the set of absolutely continuous functions f on $[0,1]$ with $f(0) = 0$ and $\int_0^1 [f'(t)]^2 dt \le 1$; thus

$$\mathcal{K} \equiv \left\{ f \in C[0,1] \;:\; f(0) = 0,\; f(t) = \int_0^t f'(s)ds,\; \int_0^1 [f'(s)]^2 ds \le 1 \right\}.$$

Theorem 12.9.4: (Strassen) For almost all ω the sequence of functions $\{t \mapsto \mathbb{Z}_n(t,\omega) : \; n = 3, 4, \ldots\}$ is relatively compact with limit set $\mathcal{K}$. That is

$$P\left(\overline{\lim}_n \|\mathbb{Z}_n - \mathcal{K}\| = 0\right) = 1 \quad \text{and}$$
$$P\left(\cap_{f \in \mathcal{K}} [\underline{\lim}_n \|\mathbb{Z}_n - f\| = 0\right) = 1.$$

We will write this conclusion symbolically as $\mathbb{Z}_n \rightsquigarrow \mathcal{K}$. By an embedding in Brownian motion we can also conclude that $\widetilde{\mathbb{Z}}_n \equiv \mathbb{S}_n(t)/\sqrt{2\log\log n}$ satisfies $\widetilde{\mathbb{Z}}_n \rightsquigarrow \mathcal{K}$.

Corollary: Let $g : C[0,1] \to \mathbb{R}$ be continuous with respect to the uniform metric $\|\cdot\|$. Then:

$$\limsup_n g(\mathbb{Z}_n) = \sup_{f\in\mathcal{K}} g(f) \quad a.s. \quad \text{and}$$
$$\liminf_n g(\mathbb{Z}_n) = \inf_{f\in\mathcal{K}} g(f) \quad a.s.$$

Example 1. Fix $t_0 \in (0,1)$, and let $g(f) = f(t_0)$. Then

$$\limsup_n g(\mathbb{Z}_n) = \limsup_n \mathbb{Z}_n(t_0) = +\sqrt{t_0} \quad a.s. \quad \text{and}$$
$$\liminf_n g(\mathbb{Z}_n) = \liminf_n \mathbb{Z}_n(t_0) = -\sqrt{t_0} \quad a.s..$$

Proof: Now

$$g(f) = f(t_0) = \int_0^{t_0} f'(s)ds = \int_0^1 1_{[0,t_0]}(s) f'(s)ds,$$

so by the Cauchy - Schwarz inequality

$$|g(f)| \le \left(\int_0^1 1_{[0,t_0]}(s)ds \right)^{1/2} \left(\int_0^1 [f'(s)]^2 ds \right)^{1/2} \le t_0^{1/2} \cdot 1$$

so

$$\sup_{f \in \mathcal{K}} g(f) = \sqrt{t_0}$$

with equality when $f(t) = (t \wedge t_0)/\sqrt{t_0}, \;\; 0 \le t \le 1$. Note that this f is linear on $[0, t_0]$ with slope $t_0^{-1/2}$, is constant on $[t_0, 1]$ at the height $\sqrt{t_0}$, and it satisfies

$$\int_0^1 [f'(s)]^2 ds = 1.$$

4. Boundary Crossing probabilities

Notation: Let $\mathbb{S}$ denote standard Brownian motion starting from 0, and let $\mathbb{U}$ denote a (standard) Brownian bridge process on $[0,1]$, and let $Z \sim N(0,1)$.

We will use the strong Markov property of $\mathbb{S}$ repeatedly.

Application 1. Let $a, b > 0$, and let $\tau_b \equiv \inf\{t > 0 : \ \mathbb{S}(t) = b\}$. Then

$$\begin{aligned} P(\sup_{0 \le t \le a} \mathbb{S}(t) \ge b) &= P(\tau_b \le a) = 2P(\mathbb{S}(a) \ge b) = P(Z \ge b/\sqrt{a}) \\ &= 2(1 - \Phi(b/\sqrt{a})). \end{aligned}$$

Proof: The first equality is obvious. The second inequality follows from a reflection argument:

$$\begin{aligned} P(\tau_b \le a) &= P(\tau_b \le a, \mathbb{S}(a) \ge b) + P(\tau_b \le a, \mathbb{S}(a) < b) \\ &= P(\tau_b \le a, \mathbb{S}(a) - \mathbb{S}(\tau_b) \ge 0) + P(\tau_b \le a, \ \mathbb{S}(a) - \mathbb{S}(\tau_b) < 0) \\ &= P(\mathbb{S}(a) \ge b) + P(\tau_b \le a) \cdot P(\mathbb{S}(\tau_b + (a - \tau_b)) - \mathbb{S}(\tau_b) < 0) \\ &= P(\mathbb{S}(a) \ge b) + P(\tau_b \le a) \cdot (1/2). \end{aligned}$$

Rearranging this last equality yields the second equality of the claim. $\square$

Application 2. Fix $b \in \mathbb{R}$. Then $\tilde{\mathbb{S}}$ defined by

$$\tilde{\mathbb{S}}(t) = \begin{cases} \mathbb{S}(t), & t < \tau_b, \\ 2b - \mathbb{S}(t), & t \geq \tau_b \end{cases}$$

is a standard Brownian motion on $[0, \infty)$.

Proof: Consider the two processes $Y(t) \equiv \mathbb{S}(t)$, $0 \leq t \leq \tau_b$, and $Z(t) \equiv \mathbb{S}(\tau_b + t) - \mathbb{S}(\tau_b) = \mathbb{S}(\tau_b + t) - b$. By the strong Markov property of BM, Z is a Brownian motion independent of Y; $-Z$ is also a Brownian motion independent of Y. Thus $(Y, Z) \overset{d}{=} (Y, -Z)$. Now $\varphi(Y, Z) = Y(t)1_{[t \leq \tau_b]} + (b + Z(t - \tau_b))1_{[t > \tau_b]}$ yields a continuous process with the same law as $\varphi(Y, -Z)$. But var$\phi(Y, Z) = \mathbb{S}$ and $\varphi(Y, -Z) = \tilde{\mathbb{S}}$. $\square$

Application 3. Let $M(t) \equiv \sup_{s \le t} \mathbb{S}(s)$. Then, for $b, y, t \ge 0$,

$$P(M(t) \ge b, \mathbb{S}(t) \le b - y) = P(\mathbb{S}(t) \ge b + y).$$

Proof: With $\tilde{\mathbb{S}}$ as in Application 2 above,

$$\begin{aligned} & P(M(t) \ge b, \mathbb{S}(t) \le b - y) = P(M(t) \ge b,\ \mathbb{S}(t) \le b - y) \\ = \ & P(M(t) \ge b,\ \tilde{\mathbb{S}}(t) \le b - y) = P(M(t) \ge b,\ 2b - \mathbb{S}(t) \le b - y) \\ = \ & P(\mathbb{S}(t) \ge b + y). \end{aligned}$$

$\square$

Application 4. Let $\tau_{a,b} \equiv \inf\{t > 0 : \mathbb{S}(t) \in (-a, b)^c\}$ for $a, b > 0$. Then for $b > 0$,

$$\begin{aligned} P(\|\mathbb{S}\|_0^1 > b) &= P(\tau_{b,b} < 1) \\ &= 4 \sum_{k=1}^{\infty} P((4k-3)b < Z < (4k-1)b) \\ &= 1 - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2\pi^2}{8b^2}\right). \end{aligned}$$

Proof: Let

$$\begin{aligned} A_+ \equiv [\|\mathbb{S}^+\| > b] = [\mathbb{S}(t) > b \ \text{ for some } \ 0 \le t \le 1], \quad \text{and} \\ A_- \equiv [\|\mathbb{S}^-\| > b] = [\mathbb{S}(t) < -b \ \text{ for some } \ 0 \le t \le 1]. \end{aligned}$$

Note that

$$[\|\mathbb{S}\|_0^1 < b] = [A_+ \cup A_-]$$

where A_+ and A_- are not disjoint, so $P(\|\mathbb{S}\| > b) \leq P(A_+) + P(A_-)$. Thus we need to consider

$$\begin{aligned} A_{+,-} &= [\mathbb{S}(t) < b \text{ for some } 0 \leq t \leq 1, \text{ then } \mathbb{S}(t) < -b \text{ for some } t > \tau_b], \\ A_{-,+} &= [\mathbb{S}(t) < -b \text{ for some } 0 \leq t \leq 1, \text{ then } \mathbb{S}(t) > b \text{ for some } t > \tau_b]. \end{aligned}$$

Note that

$$P(A_{+,-}) = P(\|\mathbb{S}^+\| > 3b),$$
$$P(A_{-,+}) = P(\|\mathbb{S}^-\| > 3b).$$

Thus ...

$$\begin{aligned}
P(\|\mathbb{S}\|_0^1 > b) &= P(A_+) - P(A_{+,-}) + P(A_{+,-,+}) \cdots - \\
&\quad + P(A_-) - P(A_{-,+}) + P(A_{-,+,-}) \cdots - \\
&= 2\left\{P(A_+) - P(A_{+,-}) + P(A_{+,-,+}) - \cdots\right\} \\
&\quad \text{by symmetry} \\
&= 2\sum_{k=1}^{\infty} (-1)^k \cdot 2P(\mathbb{S}(1) > (2k+1)b) \\
&= 4\sum_{k=1}^{\infty} P((4k-3)b < Z < (4k-1)b).
\end{aligned}$$

The last equality follows from the Poisson summation formula; cf. Feller, Volume II, pages 342-343, (X.5), and 629 - 630, IXI.5. Note that

$$\begin{aligned}
P(\|\mathbb{S}\|_0^a \geq b) &\leq P(\sup_{0\leq t\leq a} \mathbb{S}(t) \geq b) + P(\sup_{0\leq t\leq a} (-\mathbb{S}(t)) \geq b) \\
&\leq 2P(\mathbb{S}(a) \geq b) + 2P(\mathbb{S}(a) \geq b) = 4P(\mathbb{S}(a) \geq b) \\
&= 4P(Z \geq b/\sqrt{a}).
\end{aligned}$$

More boundary crossing probabilities:

Theorem 12.7.2: Let $c \ge 0$, $d > 0$. Then

$$P(\mathbb{S}(t) \ge ct + d \ \text{ for some } t > 0) = e^{-2cd},$$
$$P(|\mathbb{S}(t)| \ge ct + d \ \text{ for some } t > 0) = 2 \sum_{k=1}^{\infty} (-1)^{k+1} e^{-2k^2 cd}.$$

Proof: For any $\theta > 0$ the process

$$\begin{aligned} V(t) &\equiv \exp(\theta(\mathbb{S}(t) - \theta^2 t/2)), \quad t \ge 0, \\ &= \exp(\theta(\mathbb{S}(t) - \theta t/2)) \equiv \exp(\theta X(t)) \end{aligned}$$

is a martingale. Now define

$$\tau_{a,b} \equiv \inf\{t > 0 : \ X(t) \in (-a, b)^c\}.$$

Optional sampling gives

$$1 = EV(\tau_{a,b}) = P(X(\tau_{a,b}) = -a)e^{-\theta a} + P(X(\tau_{a,b}) = b)e^{\theta b},$$

so that

Thus

$$\begin{aligned} P(X(\tau_{a,b}) = b) &= \frac{1 - e^{-\theta a}}{e^{\theta b} - e^{-\theta a}} \\ &\to e^{-\theta b} \ \text{ if } \theta > 0 \text{ and } a \to \infty \\ &= e^{-2cd} \ \text{ if } \theta = 2c \text{ and } b = d. \end{aligned}$$

But we also have

$$\begin{aligned} P(X(\tau_{a,b}) = b) &\to P(X(t) \ge b \text{ for some } t \ge 0) \ \text{ as } a \to \infty \\ &= P(\mathbb{S}(t) - \theta t/2 \ \ge \ b \text{ for some } t \ge 0) \\ &= P(\mathbb{S}(t) \ge \theta t/2 + b \text{ for some } t \ge 0) \\ &= P(\mathbb{S}(t) \ge ct + d \text{ for some } t \ge 0) \ . \end{aligned}$$

if $\theta = 2c$ and $b = d$. $\square$

Theorem 12.7.3: Suppose that $b > 0$. Then

$$P(\|\mathbb{U}^{\pm}\| > b) = e^{-2b^2} \quad \text{for all } b > 0,$$
$$P(\|\mathbb{U}\| > b) = 2\sum_{k=1}^{\infty}(-1)^{k+1}e^{-2k^2b^2}, \quad b > 0.$$

Proof: First, $\|\mathbb{U}^{+} \overset{d}{=} \|\mathbb{U}^{-}\|$. Furthermore

$$\begin{aligned} P(\|\mathbb{U}^{+}\| > b) &= P(\mathbb{U}(t) > b \text{ for some } 0 \le t \le 1) \\ &= P\left((1-t)\mathbb{S}\left(\frac{t}{1-t}\right) > b \text{ for some } 0 \le t \le 1\right) \\ &= P(\mathbb{S}(r) > b(1+r) \quad \text{for some } r \ge 0) \\ &= e^{-2b^2} \quad \text{by Theorem 12.7.2.} \end{aligned}$$

Similarly,

$$\begin{aligned} P(\|\mathbb{U}\| > b) &= P(|\mathbb{S}(r)| > b(1+r) \quad \text{for some } r > 0) \\ &= 2\sum_{k=1}^{\infty}(-1)^{k+1}\text{exp}(-2k^2b^2). \end{aligned}$$