Math/Stat 523, Spring 2020

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Lecture 3

Friday, April 3 (or Monday April 6)

Outline

- 1: The Law of the Iterated Logarithm (LIL) for partial sums.
- 2: Skorokhod Embedding of the Uniform Empirical Process $\mathbb{U}_n$ and the uniform quantile process $\mathbb{V}_n$.
- 3: Skorokhod Embedding of Related Processes.

1. The law of the iterated logarithm

We begin by recalling the law of the iterated logarithm for partial sums:

Theorem 8.6.1: (LIL; Hartmann-Wintner; Strassen) Let $X_1, X_2, \ldots$ be i.i.d. random variables, and set $S_n = X_1 + \cdots + X_n$ for $n \geq 1$.

(a) If $E(X) = 0$ and $\sigma^2 \equiv Var[X] < \infty$, then

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \log \log n}} = +\sigma \quad \text{a.s.}$$

and

$$\liminf_{n\to\infty} \frac{S_n}{\sqrt{2n \log \log n}} = -\sigma \quad \text{a.s.}$$

(b) In fact

$$\frac{S_n}{\sqrt{2n \log \log n}} \rightsquigarrow [-\sigma, \sigma] \quad \text{a.s.}$$

That is, for a.e. ω the limit set of $S_n(\omega)/\sqrt{2n\text{loglog}n}$ is exactly $[-\sigma, \sigma]$.

(c) Conversely (Strassen), if

$$\overline{\text{lim}} \frac{|S_n|}{\sqrt{2n\text{loglog}n}} < \infty \text{ a.s.}$$

then $E(X) = 0$ and $\sigma^2 < \infty$.

Thus with $\sigma^2 = 1$ (without loss), if the X_i's are i.i.d. with mean 0 and $Var[X_1] = 1$, then

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2n\mathrm{loglog}n}} \;=\; 1 \quad \text{a.s.} \tag{1}$$

while the two LIL's for Brownian motion at ∞ and at 0 are

$$\overline{\lim}_{t\to\infty} \frac{|\mathbb{S}(t)|}{\sqrt{2t\mathrm{loglog}t}} = 1 \ \text{ a.s. } \ \text{and} \tag{2}$$

$$\overline{\lim}_{t\searrow 0} \frac{|\mathbb{S}(t)|}{\sqrt{2t\mathrm{loglog}(1/t)}} = 1 \ \text{ a.s..} \tag{3}$$

In fact the Hartmann-Wintner theorem (1) is proved in Section 8.6 of PfS only for the Gaussian case. Strassen's proof of (1) involves proving (2), and then proving (1) via embedding in Brownian motion as follows:

Notation: Let $\mathbb{S}$ be standard Brownian motion, and define stopping times $T_1, T_2, \ldots$ having mean 1 (and $T_0 \equiv 0$), so that

$$X_k \equiv \mathbb{S}(\tau_{k-1}, \tau_k] = \mathbb{S}(\tau_k) - \mathbb{S}(\tau_{k-1}) \text{ are i.i.d. as } F$$

where $\tau_k \equiv T_0 + \cdots + T_k$ for $k \geq 0$. Then the partial sums are

$$S_n \equiv \sum_{k=1}^{n} X_k \equiv \mathbb{S}(\tau_n) = \mathbb{S}(n) + \mathbb{S}(\tau_n) - \mathbb{S}(n).$$

Proof: We suppose that (2) holds. Use this to show that (1) holds by showiing

$$\frac{|\mathbb{S}(\tau_n) - \mathbb{S}(n)|}{\sqrt{2n\text{loglog}n}} \to_{a.s.} 0, \quad \text{or}$$
$$\frac{|\mathbb{S}(\tau_{\lfloor t \rfloor}) - \mathbb{S}(t)|}{\sqrt{2t\text{loglog}t}} \to_{a.s.} 0. \tag{4}$$

We will also show that

$$\sup_{s \leq t} \frac{|\mathbb{S}(\tau_{\lfloor s \rfloor}) - \mathbb{S}(s)|}{\sqrt{t}} \to_p 0. \tag{5}$$

Lemma: For a Brownian motion $\mathbb{S}$ we have

$$\lim_{r\searrow 1}\limsup_{t\to\infty}\sup_{t\le u\le rt}\frac{|\mathbb{S}(u)-\mathbb{S}(t)|}{\sqrt{2t\text{loglog}t}}=0 \quad \text{a.s..}$$

Proof. Let $h(t)\equiv\sqrt{2t\text{loglog}t}$. It suffices to how that

$$\lim_{r\searrow 1}\overline{\lim}_n\sup_{r^n\le t\le r^{n+1}}\frac{|\mathbb{S}(t)-\mathbb{S}(r^n)|}{h(r^n)}=0 \quad \text{a.s. .}$$

But then, using Mills' ratio for the second inequality,

$$
\begin{aligned}
P\left(\sup_{r^n \le t \le r^{n+1}} \frac{|\mathbb{S}(t)-\mathbb{S}(r^n)|}{ch(r^n)} \ge 1\right) &\equiv P(A_n)\\
&\le 2P\left(\mathbb{S}(r^{n+1})-\mathbb{S}(r^n) > ch(r^n)\right)\\
&= 2P\left(\mathbb{S}(r^n(r-1)) > ch(r^n)\right)\\
&= 2P\left(Z > \frac{c}{\sqrt{r-1}}\sqrt{2\log(n\log r)}\right)\\
&\le \frac{2}{\sqrt{2\pi}}\sqrt{\frac{r-1}{2c^2\log(n\log r)}}\exp\left(-\frac{c^2}{r-1}\log n\right)\\
&\le \frac{4}{\sqrt{2\pi}}\sqrt{\frac{r-1}{2c^2\log n}}\, n^{-c^2/(r-1)} \quad \text{for } n \text{ large.}
\end{aligned}
$$

Now fix $c > 0$ and let $r > 1$ satisfy $c^2/(r-1) \ge 1+\delta$ for some $\delta > 0$. Then $\sum_n P(A_n) < \infty$ and $P(A_n \text{ i.o.}) = 0$. $\square$

The Lemma implies that (4) holds. Now we show that (5) holds. Note that

$$\frac{\delta_t}{t} \equiv \frac{1}{t} \sup_{s \le t} |\tau_{\lfloor s \rfloor} - s| \to 0.$$

Let

$$w(f; t, h) \equiv \sup_{r,s \le t, |r-s| \le h} |f(r) - f(s)|.$$

Fix t, h, $\epsilon > 0$. By the scaling property of Brownian motion,

$$\begin{aligned} & P\left(t^{-1/2} \sup_{s \le t} |\mathbb{S}(\tau_{\lfloor s \rfloor}) - \mathbb{S}(s)| > \epsilon\right) \\ & \quad \le \; P\left(w(\mathbb{S}, t + th, th) > \epsilon\sqrt{t},\ \delta_t \le th\right) + P(\delta_t \le th) \\ & \quad = \; P\left(w(\mathbb{S}, 1 + h, h) > \epsilon,\ \delta_t \le th\right) + P(t^{-1}\delta_t > h) \\ & \quad \to \; 0 \quad \text{as} \quad t \to \infty, \quad \text{and then} \quad h \searrow 0. \end{aligned}$$

This proves (5). $\square$

These developments lead to Strassen's functional LIL which will be discussed (at least briefly) in Lecture 4.

Question: Suppose that $S_n^{(1)}$ and $S_n^{(2)}$ are two independent partial sums based on $\{X_k^{(1)}: \ k \geq 1\}$ and $\{X_k^{(2)}: \ k \geq 1\}$ where the $X_k^{(1)}$'s and $X_k^{(2)}$'s are all mean zero and variance 1. Thus by the 1-dimensional LIL the a.s. limit set of each coordinate is $[-1, 1]$. What is the limit set for the pairs $(S_n^{(1)}, S_n^{(2)})$ in $\mathbb{R}^2$? Is it the (unit box) $[-1, 1] \times [-1, 1]$? Or is it something smaller?

2: Skorokhod Embedding of the Uniform Empirical Process $\mathbb{U}_n$

(This section is based on PfS, section 12.10, pages 329-336.)

Let $\xi_{n1}, \ldots, \xi_{nn}$ be i.i.d. Uniform$(0,1)$. Their empirical d.f. $\mathbb{G}_n$ is defined by

$$\begin{aligned} \mathbb{G}_n(t) &\equiv \frac{1}{n}\sum_{i=1}^{n} 1_{[0,t]}(\xi_{ni}) \quad \text{for} \quad 0 \le t \le 1 \\ &= k/n \quad \text{for} \quad \xi_{n:k} \le t < \xi_{n:k+1} \quad \text{and} \quad 0 \le k \le n, \end{aligned}$$

where $0 \equiv \xi_{n:0} \le \xi_{n:1} \le \cdots \le \xi_{n:n} \le \xi_{n:n+1} \equiv 1$ are the order statistics. Note that $n\mathbb{G}_n(t) \sim$ Binomial(n,t). The Glivenko - Cantelli theorem shows that

$$\|\mathbb{G}_n - I\| \to_{a.s.} 0 \quad \text{(even for the present triangular array of } \xi_{n,k}\text{'s)}.$$

The uniform empirical process is defined by

$$\mathbb{U}_n(t) \equiv \sqrt{n}(\mathbb{G}_n(t) - t) = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}[1_{[\xi_{n:k} \le t]} - t] \quad \text{for} \quad 0 \le t \le 1.$$

Then with $\mathbb{G}_n^{-1}(t) \equiv \inf\{x \in [0,1] : \ \mathbb{G}_n(x) \ge t\}$ we have

$$\mathbb{G}_n^{-1}(t) = \xi_{n:k} \quad \text{for} \quad (k-1)/n < t \le k/n$$

and $\mathbb{G}_n^{-1}(0) = 0$. The *uniform quantile process* $\mathbb{V}_n$ is

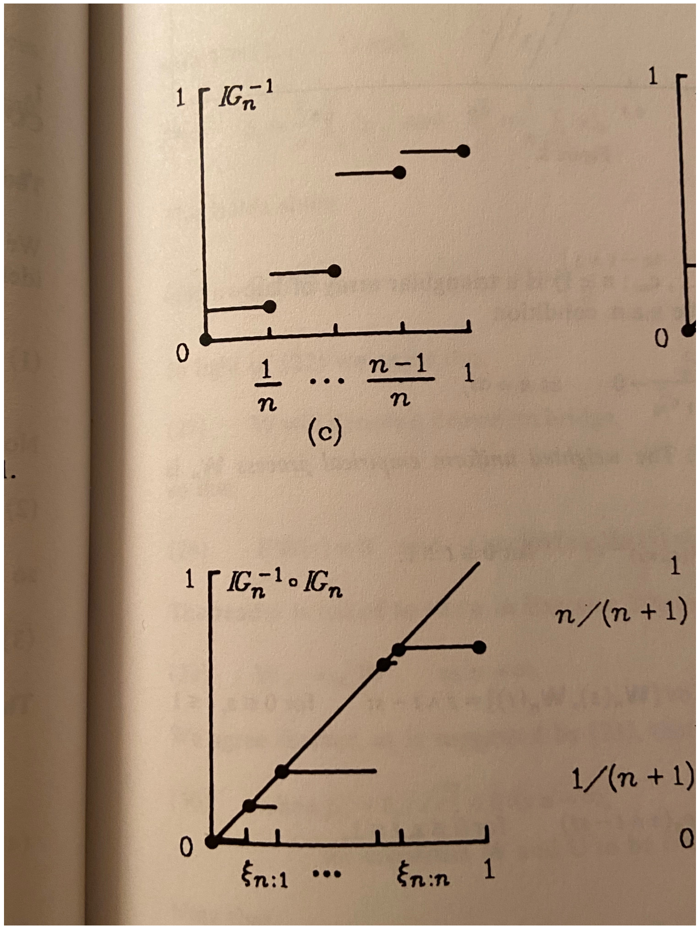
$$\mathbb{V}_n(t) = \sqrt{n}(\mathbb{G}_n^{-1}(t) - t) \quad \text{for} \quad 0 \le t \le 1.$$

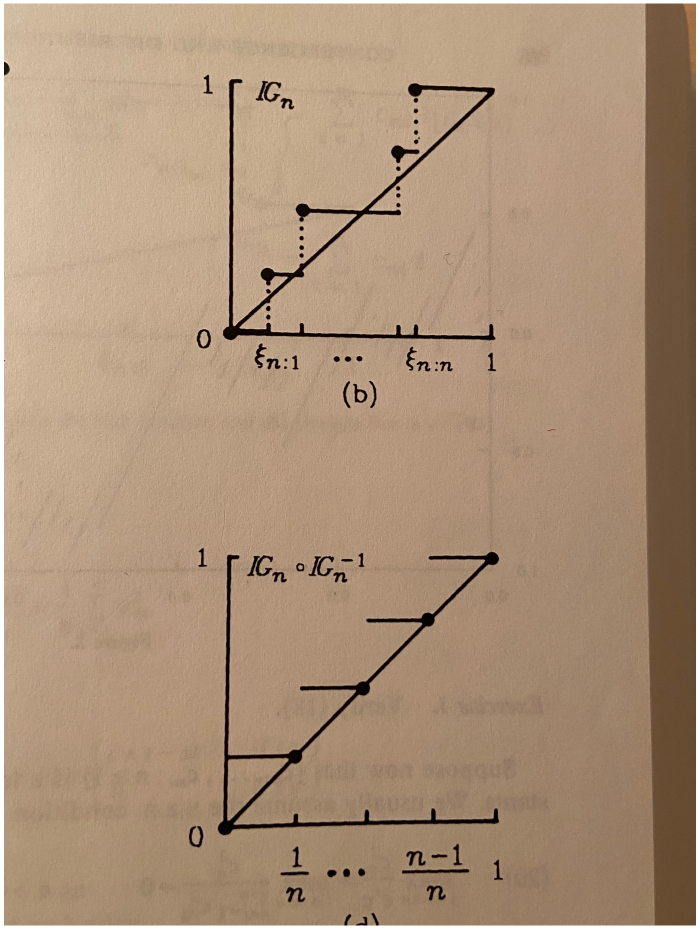
Here are the key identities relating $\mathbb{U}_n$ and $\mathbb{V}_n$:

$$\begin{aligned} \mathbb{U}_n &= -\mathbb{V}_n(\mathbb{G}_n) + \sqrt{n}(\mathbb{G}_n^{-1} \circ \mathbb{G}_n - I) \quad \text{on} \quad [0,1] \\ \mathbb{V}_n &= -\mathbb{U}_n(\mathbb{G}_n^{-1}) + \sqrt{n}(\mathbb{G}_n \circ \mathbb{G}_n^{-1} - I) \quad \text{on} \quad [0,1]. \end{aligned}$$

Note that:

- $\|\mathbb{G}_n^{-1} - I\| = \|\mathbb{G}_n - I\| \to_p 0$.
- $\|\mathbb{G}_n \circ \mathbb{G}_n^{-1} - I\| = 1/n \to 0$.
- $\|\mathbb{G}_n^{-1} \circ \mathbb{G}_n - I\| = \max_{1 \le k \le n+1} \delta_{n:k}$, $\quad \delta_{n:k} \equiv \xi_{n:k} - \xi_{n:k-1}$.


1
$\mathbb{G}_n^{-1}$
0
$\frac{1}{n}$ … $\frac{n-1}{n}$
1
(c)
1
$\mathbb{G}_n^{-1} \circ \mathbb{G}_n$
0
$\xi_{n:1}$ … $\xi_{n:n}$
1
1
$n/(n+1)$
$1/(n+1)$
0


1
$\mathbb{G}_n$
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1
(b)
1
$\mathbb{G}_n \circ \mathbb{G}_n^{-1}$
0
$\frac{1}{n}$
$\cdots$
$\frac{n-1}{n}$
1

Our goal is to prove the following theorem:

Theorem 12.10.1: (Breiman and Brillinger) We can define Brownian bridges $\mathbb{U} = -\mathbb{V}$ in such a way that

$$\|\mathbb{U}_n - \mathbb{U}\| \to_p 0 \quad \text{and} \quad \|\mathbb{V}_n - V\| \to_p 0.$$

Proof. We start with $\mathbb{V}_n$. We will represent the uniform$(0, 1)$ rv's as normed sums of independent Exponential(1) rv's. Let $F(x) = 1 - \exp(-(x+1))$ for $x \geq -1$, so that F has zero mean and variance 1. If $X \sim F$, then $X + 1 \sim$ Exponential(1). According to Skorokhod's embedding theorem, there exist row-independent rv's $X_{n1}, \ldots, X_{nn}$ with df F such that the partial sum process $\mathbb{S}_n$ of the nth row satisfies $\|\mathbb{S}_n - \mathbb{S}\| \to_p 0$ for some Brownian motion $\mathbb{S}$.

Now define

$$\eta_{nk} \equiv k + X_{n1} + \cdots + X_{nk} \quad \text{and} \quad \xi_{n:k} \equiv \eta_{nk}/\eta_{n,n+1}$$

for $1 \le k \le n+1$. It is an easy exercise to show that these ξ_{nk}'s are distributed as n row-independent Uniform$(0,1)$ order statistics. The basic identity relating $\mathbb{V}_n$ to $\mathbb{S}_n$ is

$$\begin{aligned}
\mathbb{V}_{n-1}\left(\frac{k}{n-1}\right) &= \sqrt{n-1}\left\{\frac{\eta_k}{\eta_n} - \frac{k}{n-1}\right\} \\
&= \frac{n}{\eta_n}\sqrt{\frac{n-1}{n}}\left\{\frac{\eta_k - k}{\sqrt{n}} - \frac{k}{n}\cdot\frac{\eta_n - n}{\sqrt{n}}\right\} \\
&\qquad - \sqrt{n-1}\left\{\frac{k}{n-1} - \frac{k}{n}\right\} \\
&= \frac{n}{\eta_n}\sqrt{\frac{n-1}{n}}\left\{\mathbb{S}_n(\frac{k}{n}) - \frac{k}{n}\mathbb{S}_n(1)\right\} - \frac{1}{\sqrt{n-1}}\cdot\frac{k}{n} \quad (6)
\end{aligned}$$

Thus for $0 \le t \le 1$,

$$\begin{aligned} \mathbb{V}_{n-1}(t) &= \frac{n}{\eta_n}\sqrt{\frac{n-1}{n}}\,\{\mathbb{S}_n(I_n(t)) - I_n(t)\mathbb{S}_n(1)\} \\ &\quad -\frac{1}{\sqrt{n-1}} I_n(t) \end{aligned} \tag{7}$$

where $I_n(t) \equiv k/n$ for $(k-1)/(n-1) < t \le k/(n-1)$ and $1 \le k \le n-1$ with $I_n(0) = 0$ satisfies $\|I_n - I\| \to 0$. Furthermore $\eta_n/n \to_p 1$ by the WLLN. Thus

$$\|\mathbb{V}_n - \mathbb{V}\| \to_p 0 \quad \text{for} \quad \mathbb{V} \equiv \mathbb{S} - I\,\mathbb{S}(1) \tag{8}$$

follows from the identity (7), $\|I_n - I\| \to 0$, and the fact that

$$\begin{aligned} \|\mathbb{S}_n(I_n) - \mathbb{S}\| &\le \|\mathbb{S}_n(I_n) - \mathbb{S}(I_n)\| + \|\mathbb{S}(I_n) - \mathbb{S}\| \\ &\le \|\mathbb{S}_n - \mathbb{S}\| + \|\mathbb{S}(I_n) - \mathbb{S}\| \to_p 0. \end{aligned}$$

Now we need to carry the result (8) over to the uniform empirical process $\mathbb{U}_n$.

First note that all the sample paths of $\mathbb{V}$ are continuous, and the maximum jump size of $|\mathbb{V}_n - \mathbb{V}|$ is bounded above by $\sqrt{n}\max_{1\le i\le n+1}\delta_{n:i}$, so $\|\mathbb{V}_n - \mathbb{V}\| \to_p 0$ and the first equality in the next display imply that

$$\sqrt{n}\max_{1\le i\le n+1}\delta_{n:i} = \sqrt{n}\|\mathbb{G}_n^{-1}\circ\mathbb{G}_n - I\| \to_p 0 \quad \text{as} \quad n\to\infty.$$

Thus with $\mathbb{U} \equiv -\mathbb{V}$

$$\begin{aligned}
\|\mathbb{U}_n - \mathbb{U}\| &= \|-\mathbb{V}_n(\mathbb{G}_n) + \sqrt{n}(\mathbb{G}_n^{-1}\circ\mathbb{G}_n - I) + \mathbb{V}\| \\
&\le \|\mathbb{V}_n(\mathbb{G}_n) - \mathbb{V}(\mathbb{G}_n)\| + \|\mathbb{V}(\mathbb{G}_n) - \mathbb{V}\| + \sqrt{n}\|\mathbb{G}_n^{-1}\circ\mathbb{G}_n - I\| \\
&\le \|\mathbb{V}_n - \mathbb{V}\| + \|\mathbb{V}(\mathbb{G}_n) - \mathbb{V}\| + \sqrt{n}\max_{1\le i\le n+1}\delta_{n:i} \\
&\to_p 0
\end{aligned}$$

where we also used $\|\mathbb{G}_n - I\| \to_p 0$.

Note that $\mathbb{G}_n^{-1}(t) = \xi_{n:k}$ for $(k-1)/n < t \le k/n$, so

$$\begin{aligned}
\mathbb{G}_n \circ \mathbb{G}_n^{-1}(t) &= \mathbb{G}_n(\xi_{n:k}) \quad \text{for} \quad (k-1)/n < t \le k/n \\
&= k/n, \quad \text{for} \quad (k-1)/n < t \le k/n.
\end{aligned}$$

Note that this is completely deterministic, being a step function above the identity function at distance at most $1/n$.
Thus $\|\mathbb{G}_n \circ \mathbb{G}_n^{-1} - I\| \le 1/n$.

Similarly, note that $\mathbb{G}_n(t) = k/n$ for $\xi_{n:k} \le t < \xi_{n:k+1}$, so

$$\begin{aligned}
\mathbb{G}_n^{-1} \circ \mathbb{G}_n(t) &= \mathbb{G}_n^{-1}(k/n) \quad \text{for} \quad \xi_{n:k} \le t < \xi_{n:k+1}, \quad k = 0, \ldots, n \\
&= \xi_{n:k}, \quad k = 0, \ldots, n.
\end{aligned}$$

Thus

$$\begin{aligned}
\|\mathbb{G}_n^{-1} \circ \mathbb{G}_n - I\| &= \max_{1 \le k \le n+1} (\xi_{n:k} - \xi_{n:k-1}) \\
&\equiv \max_{1 \le k \le n+1} \delta_{n:k}.
\end{aligned}$$