Math/Stat 523, Spr 2020

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Lecture 2

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Outline

- 1: The stopping time $\tau_{a,b}$ for Brownian motion $\mathbb{S}$
- 2: Embedding one random variable X in $\mathbb{S}$.
- 3: Embedding the partial sum process $\mathbb{S}_n$ in $\mathbb{S}$.

1. The stopping time $\tau_{a,b}$

Let $a, b > 0$ and let $\tau \equiv \tau_{a,b} \equiv \inf\{t > 0 : \mathbb{S}(t) \in (-a, b)^c\}$.
Recall our results from winter quarter concerning τ and $\mathbb{S}(\tau)$.

- (2) $E\mathbb{S}(\tau) = 0$.
- (3) $P(\mathbb{S}(\tau) = -a) = \frac{b}{a+b}$ and $P(\mathbb{S}(\tau) = b) = \frac{a}{a+b}$.
- (4) $E(\tau) = ab = E\mathbb{S}^2(\tau)$ and $E(\tau^2) \le 4ab(a+b)$.
- (5) $E(\tau^r) \le r\Gamma(r)2^{2r}E\mathbb{S}^{2r}(\tau) \le r\Gamma(r)2^{2r}ab(a+b)^{2r-2}$
  for all $r \ge 1$.

2. Embedding one random variable X in $\mathbb{S}$

Theorem (Skorokhod) Suppose that X is a random variable with d.f. F having mean 0 and variance $Var[X] \equiv \sigma^2 \in [0, \infty]$. Then there is a stopping time τ such that the stopped random variable $\mathbb{S}(\tau)$ is distributed as X; that is

$$\mathbb{S}(\tau) \stackrel{d}{=} X.$$

Moreover,

$$E\tau = Var[X] \quad \text{and} \quad E\tau^2 \leq 16E[X^4],$$

and, for any $r \geq 1$ we have

$$E(\tau^r) \leq K_r E[X^{2r}] \quad \text{with} \quad K_r \equiv r\Gamma(r)2^{4r-2}.$$

Proof. For degenerate F just let $\tau \equiv 0$. Thus suppose F is non-degenerate. Let (A, B) be independent of $\mathbb{S}$ with joint d.f. H given by

$$dH(a,b) = \frac{(a+b)}{EX^+} dF(-a) dF(b) \quad \text{for} \quad a \in [0, \infty),\ b \in (0, \infty).$$

The procedure is to observe $(A, B) = (a, b)$ distributed according to H, and then to observe $\tau_{a,b}$ calling the result τ. (Clearly $\tau_{a,b} = 0$ if $A = a = 0$ is observed.) Note that $[\tau \le t]$ can be determined by (A, B) and $\{\mathbb{S}(s), 0 \le s \le t\}$, and hence is an event in $\mathcal{A}_t \equiv \sigma[A, B, \mathbb{S}(s) :\ 0 \le s \le t]$. Then, for $t \ge 0$,

$$
\begin{aligned}
P(\mathbb{S}(\tau) > t) &= E\{P(\mathbb{S}(\tau) > t | A, B)\} \\
&= \int_{[0,\infty)} \int_{(0,t]} 0 \cdot dH(a,b) + \int_{[0,\infty)} \int_{(t,\infty)} \frac{a}{a+b} dH(a,b) \\
&\qquad \text{by} \quad (3) \\
&= \int_{(t,\infty)} \int_{[0,\infty)} \frac{a}{EX^+} dF(-a) dF(b) = \int_{(t,\infty)} \frac{E(X^-)}{E(X^+)} dF(b) \\
&= 1 - F(t)
\end{aligned}
$$

since $0 = E(X) = E(X^+) - E(X^-)$ with F non-degenerate implies $E(X^+) = E(X^-)$. Similarly, for $t \geq 0$,

$$
\begin{aligned}
P(\mathbb{S}(\tau) \leq -t) &= E\{P(\mathbb{S}(\tau) \leq -t | A, B)\} \\
&= \int_{0,t)} \int_{(0,\infty)} 0 \cdot dH(a,b) + \int_{[t,\infty)} \int_{(0,\infty)} \frac{b}{a+b} dH(a,b) \\
&= +\int_{[t,\infty)} \int_{(0,\infty)} \frac{b}{EX^+} dF(b) dF(-a) = \int_{[t,\infty)} dF(-a) \\
&= F(-t).
\end{aligned}
$$

Thus $\mathbb{S}(\tau) \overset{d}{=} X$. Furthermore,

$$\begin{aligned} E(\tau) &= E\{E[\tau|A,B]\} = E\{E[\mathbb{S}^2(\tau)|A,B]\} \\ &= E\{\mathbb{S}^2(\tau)\} = E[X^2] = Var[X]. \end{aligned}$$

Note that

$$(a+b)^{2r-1} \le 2^{2r-2}[a^{2r-1} + b^{2r-1}]$$

by the C_r-inequality. Therefore

$$\begin{aligned} E[\tau^r] &= E\{E[\tau^r|A,B]\} \\ &\le 2^{2r} r\Gamma(r) E[AB(A+B)^{2r-2}] \\ &= 2^{2r} r\Gamma(r) E[AB(A+B)^{2r-1}/(A+B)] \\ &\le r\Gamma(r) 2^{4r-2} E\left(\frac{B}{A+B}A^{2r} + \frac{A}{A+B}B^{2r}\right) \\ &= K_r E\{E[\mathbb{S}^{2r}(\tau)|A,B]\} = K_r E[\mathbb{S}^{2r}(\tau)] = K_r E[X^{2r}]. \end{aligned}$$

Embedding the partial sum process $\mathbb{S}_n$ in $\mathbb{S}$

The partial sum process

Let $X_{n,1}, \ldots, X_{n,n}$ be row - independent rv's having a common d.f. F with mean 0 and variance 1. Let $X_{n,0} \equiv 0$. We define the partial sum process on $(D, \mathcal{D})$ by

$$\mathbb{S}_n(t) = \frac{1}{\sqrt{n}} \sum_{i=0}^{\lfloor nt \rfloor} X_{n,i} = \frac{1}{\sqrt{n}} \sum_{i=0}^{k} X_{ni} \tag{1}$$

for $k/n \le t < (k+1)/n$, $0 \le k \le n$, (or for all $k \ge 0$ in case the nth row is $(X_{n1}, X_{n2}, \ldots)$). Note that

$$\begin{aligned} Cov(\mathbb{S}_n(s), \mathbb{S}_n(t)) &= \sum_{i=1}^{\lfloor ns \rfloor} \sum_{i=1}^{\lfloor nt \rfloor} Cov[X_{ni}, X_{nj}]/n \\ &= \lfloor n(s \wedge t) \rfloor / n \quad \text{for} \quad 0 \le s, t \le 1 \end{aligned}$$

for the greatest integer function $\lfloor \cdot \rfloor$. We have already proved that $\mathbb{S}_n \to_d \mathbb{S}$; we will now establish a stronger result.

Embedding the partial sum process $\mathbb{S}_n$ in $\mathbb{S}$

Notation: Let $\{\mathbb{S}(t) : \; t \geq 0\}$ denote a Brownian motion on $(C_\infty, \mathcal{C}_\infty)$. Then

$$\mathbb{Z}_n(t) \equiv \sqrt{n}\mathbb{S}(t/n) \quad \text{for } t \geq 0 \quad \text{is also a Brownian motion.}$$

By using the Skorokhod embedding technique of the previous section repeatedly on the Brownian motion $\mathbb{Z}_n$, we may guarantee that for appropriate stopping times $\tau_{n1}, \ldots, \tau_{nn}$ (with all $\tau_{n0} = 0$) we obtain that

$$\begin{aligned} X_{nk} &\equiv \mathbb{Z}_n(\tau_{n,k-1}, \tau_{n,k}] \\ &\equiv \mathbb{Z}_n(\tau_{n,k}) - \mathbb{Z}_n(\tau_{n,k-1}) \quad \text{for} \quad 1 \leq k \leq n, \end{aligned}$$

are i.i.d. rv's with distribution function F having mean 0 and variance 1. Let $\mathbb{S}_n$ denote the partial sum process of these X_{nk}'s.

Then for $t \geq 0$ we have

$$\begin{aligned} \mathbb{S}_n(t) &= \frac{1}{\sqrt{n}} \mathbb{Z}_n(\tau_{n,\lfloor nt \rfloor}) = \mathbb{S}\left(\frac{\tau_{n,\lfloor nt \rfloor}}{n}\right) \\ &= \mathbb{S}\left(\frac{1}{n}\sum_{k=1}^{\lfloor nt \rfloor} T_{nk}\right) = \mathbb{S}(I_n(t)) \end{aligned}$$

with $T_{nk} \equiv (\tau_{nk} - \tau_{n,k-1})$ and $I_n \equiv n^{-1}\tau_{n,\lfloor nt \rfloor}$. Note that:

$X_{n1}, \ldots, X_{nn}$ are i.i.d. F in each row,

$T_{n1}, \ldots, T_{nn}$ are i.i.d. with means $= 1 = Var(X)$, in each row.

$E(T_{nk}^r) \leq K_r \cdot E|X_{nk}|^{2r}$, with $K_r \equiv r\Gamma(r)2^{4r-2}$.

With this notation we have the following theorem:

Theorem. (Skorokhod's embedding theorem). The partial sum process $\mathbb{S}_n$ on $(D, \mathcal{D})$ of row- independent rv's with common distribution F as formed above satisfies:

$$\|\mathbb{S}_n - \mathbb{S}\| \equiv \sup_{0 \le t \le 1} |\mathbb{S}_n(t) - \mathbb{S}(t)| \to_p 0 \quad \text{as} \quad n \to \infty. \tag{2}$$

Proof: Let I denote the identity function. Suppose we now show

$$\|I_n - I\|_0^1 \equiv \sup_{0 \le t \le 1} \left| \frac{\tau_{n, \lfloor nt \rfloor}}{n} - t \right| \to_p 0.$$

Then on any subsequence n' where $\to_p 0$ in the last display can be replaced by $\to_{a.s.} 0$, the continuity of the paths of $\mathbb{S}$ will yield

$$\|\mathbb{S}_{n'}(\cdot) - \mathbb{S}(\cdot)\| = \|\mathbb{S}(I_{n'}) - \mathbb{S}\| \to_{a.s.} 0,$$

and thus (2) will follow.

The WLLN gives

$$I_n(t) = \tau_{n,\lfloor nt \rfloor}/n \to_p t \quad \text{for any fixed} \quad t \in (0,1].$$

By diagonalization we can extract from any subsequence a further subsequence n' on which

$$I_{n'}(t) = \tau_{n,\lfloor nt \rfloor}/n \to_{a.s.} t \quad \text{for all rational} \quad t.$$

But since all functions involved are monotone, and since the limit function is continuous, thus the conclusion (2) holds. □

The embedding for $C[0,\infty)$

In the proof just given the conclusion can trivially be replaced by

$$\sup_{0\le t\le m} |\mathbb{S}_n(t) - \mathbb{S}(t)| \to_p 0$$

for any $m \in \mathbb{N}$. By appealing to PfS Exercise 12.1.6, page 299, we may conclude that

$$\rho_\infty(\mathbb{S}_n, \mathbb{S}) \to_p 0$$

where, for functions x and y on $[0,\infty)$,

$$\rho_\infty(x,y) \equiv \sum_{k=1}^{\infty} 2^{-k} \frac{\rho_k(x,y)}{1+\rho_k(x,y)}$$

and $\rho_k(x,y) \equiv \sup_{0\le t\le k} |x(t) - y(t)|$.

Let $g : (D, \mathcal{D}) \to (\mathbb{R}, \mathcal{B})$ and let Δ_g denote the set of all $x \in D$ for which g is not $\|\cdot\|-$continuous at x. If there exists a set $\Delta \in \mathcal{D}$ having $\Delta_g \subset \Delta$ and $P(\mathbb{S} \in \Delta) = 0$, then we say that g is *a.s.* $\|\cdot\|-$*continuous* with respect to the process $\mathbb{S}$.

Theorem 12.8.3 (Donsker) Let $g : (D, \mathcal{D}) \to (\mathbb{R}, \mathcal{B})$ denote an a.s. $\|\cdot\|-$continuous mapping that is $\mathcal{D}-$measurable. Then $g : (\Omega, \mathcal{A}, P) \to (\mathbb{R}, \mathcal{B})$, and both

$$g(\mathbb{S}_n) \to_p g(\mathbb{S}) \quad \text{as } n \to \infty \quad \text{for the constructed } \mathbb{S}_n, \quad \text{and}$$
$$g(\mathbb{S}_n) \to_d g(\mathbb{S}) \quad \text{as } n \to \infty \quad \text{for any } \mathbb{S}_n \text{ having the same distribution.}$$

Example 1. For $x \in C[0, 1]$, let

$$g(x) = \lambda\{t \in [0, 1] : \ x(t) > 0\} = \int_0^1 1_{(0,\infty)}(x(t))dt$$

where λ denotes Lebesgue measure on $[0, 1]$.

Then g is not $\|\cdot\|$-continuous on $C[0,1]$. But it is a.s. $\|\cdot\|-$continuous with respect to $\mathbb{S}$; see Billingsley (1968), Appendix II, pp. 230 ff. Thus $g(\mathbb{S}_n) \to_d g(\mathbb{S}) \equiv U$. It turns out that U has the *arcsin* (= Beta(1/2, 1/2)) distribution, $P(U \le u) = (2/\pi)\arcsin(\sqrt{u})$ for $0 \le u \le 1$, with density $f_U(u) = \pi^{-1}(u(1-u))^{-1/2} 1_{(0,1)}(u)$.

In fact, if

$$
\begin{aligned}
h_1(x) &\equiv \sup\{t \in [0,1] : \ x(t) = 0\}, \\
h_2(x) &\equiv \lambda\{t \in [0,1] : \ x(t) > 0\} \equiv g(x), \\
h_3(x) &\equiv \lambda\{t \in [0, h_1(x)] : \ x(t) > 0\}, \\
h_4(x) &\equiv x(1),
\end{aligned}
$$

Billingsley (1968) shows that

$$
\begin{aligned}
(h_1(\mathbb{S}_n), h_2(\mathbb{S}_n), h_3(\mathbb{S}_n), \mathbb{S}_n(1)) &\to_d (h_1(\mathbb{S}), h_2(\mathbb{S}), h_3(\mathbb{S}), \mathbb{S}(1)) \\
&\equiv (T, U, V, \mathbb{S}(1)),
\end{aligned}
$$

and he also finds an explicit formula for the joint density of $(T, V, \mathbb{S}(1))$, and shows how this determines the joint distribution of $(T, U, V, \mathbb{S}(1))$.