Math/Stat 523, Spring 2020

Jon A. Wellner

Lecture 16

Wednesday, May 27, Friday, May 29

Outline, Stochastic Calculus, Chapter 4

A. Parabolic equations

- **4.1** The heat equation
- **4.2** The inhomogeneous equation
- **4.3** The Feynman - Kac formula

B. Elliptic equations

- **4.4** The Dirichlet problem
- **4.5** Poisson's equation
- **4.6** The Schrödinger equation

C. Applications to Brownian motion

- **4.7** Exit distributions for the ball
- **4.8** Occupation times for the ball
- **4.9** Laplace transforms, arcsine law

4A. Parabolic equations

This first part of Chapter 4 shows how Brownian motion can be used to construct solutions of several of the classical differential equations:

$$\begin{aligned} u_t &= \frac{1}{2}\Delta u \\ u_t &= \frac{1}{2}\Delta u + g \\ u_t &= \frac{1}{2}\Delta u + cu \end{aligned}$$

in $(0,\infty) \times \mathbb{R}^d$ subject to the boundary condition: u continuous at each point of $\{0\} \times \mathbb{R}^d$ and $u(0,x) = f(x)$ for $x \in \mathbb{R}^d$. Here

$$\Delta u = \frac{\partial^2 u_1}{\partial x_1^2} + \cdots + \frac{\partial^2 u_d}{\partial x_d^2}$$

and by a classical solution, we mean one that has enough derivatives for the equation to make sense. That is, $u \in C^{1,2}$, the functions that have one continuous derivative wrt t and two with respect to each of the x_i's. The continuity in u in the boundary condition is needed to establish a connection between the equation which holds in $(0, \infty) \times \mathbb{R}^d$ and $u(0, x) = f(x)$ which holds on $\{0\} \times \mathbb{R}^d$. Note that the boundary condition cannot possibly hold unless $f : \mathbb{R}^d \mapsto \mathbb{R}$ is continuous.

We will see that the solutions to these equations are (under suitable assumptions) given by

$$
\begin{aligned}
&E_x f(B_t) \\
&E_x \left(f(B_t) + \int_0^t g(t-s, B_s) dB_s \right) \\
&E_x \left(f(B_t) \exp\left(\int_0^t c(t-s, B_s) ds \right) \right).
\end{aligned}
$$

In words, the solutions can be described as follows:

(i) To solve the heat equation, run a Brownian motion and let $u(t,x) = E_x f(B_t)$.

(ii) To introduce a term g, add the integral of g along the path.

(iii) To introduce cu, multiply $f(B_t)$ by $m_t = \exp(\int_0^t c(t-s, B_s)ds)$ before taking expected values. Here we think of the Brownian particle as having mass 1 and time 0 and changing mass according to $m_s' = c(t-s, B_s)m_s$, and when we take expected values, we take the particle's mass into account.

4.1. The Heat Equation

Here we consider the basic heat equation:
(1.1a) $u \in C^{1,2}$ and $u_t = (1/2)\Delta u$ in $(0,\infty) \times \mathbb{R}^d$.
(1.1b) u is continuous at each point of $\{0\} \times \mathbb{R}^d$ and
$u(0,x) = f(x)$.

The name of this equation is due to the fact that if the units of measurement are chosen suitably then the solution $u(t,x)$ gives the temperature at the point $x \in \mathbb{R}^d$ at time t when the temperature profile at time 0 is given by $f(x)$. The first step in solving (1.1) is to find a local martingale.

Theorem (1.2): If u satisfies (1.1a), then $M_s = u(t-s, B_s)$ is a local martingale on $[0,t)$.

Proof: Applying Itô's formula, (10.2) in Chapter 2 to $u(x_0,\ldots,x_d)$ with

$X_s^0 = t - s$, and $X_s^i = B_s^i$ for $1 \le i \le d$ gives

$$\begin{aligned} u(t-s, B_s) - u(t, B_0) &= \int_0^s -u_t(t-r, B_r)dr + \int_0^s \nabla u(t-r, B_r) \cdot dB_r \\ &\quad + \frac{1}{2}\int_0^s \Delta u(t-r, B_r)dr. \end{aligned}$$

To obtain this, we used $dX_r^0 = -dr$ and X_r^0 has bounded variation, while the X_r^i with $1 \le i \le d$ are independent Brownian motions, so

$$\langle X^i, X^j \rangle_r = \begin{cases} r, & 1 \le i = j \le d, \\ 0, & \text{otherwise.} \end{cases}$$

(1.2) follows easily from the Itô formula above since $-u_t + (1/2)\Delta u = 0$ and the second term on the right side is a local martingale.

Now for a uniqueness theorem.

Theorem (1.3): If there is a solution of (1.1) that is bounded then it must be $v(t,x) \equiv E_x f(B_t)$.

Here $\equiv$ means that the last equation defines v. We will always use u for a generic solution of the equation and v for our special solution.

Proof: If we assume that u is bounded, then M_s for $0 \le s < t$, is a bounded martingale. The martingale convergence theorem implies that

$$M_t \equiv \lim_{s \nearrow t} M_s \qquad \text{exists a.s.}$$

If u satisfies (1.1b), this limit must be $f(B_t)$. Since M_s is uniformly integrable, it follows that

$$u(t,x) = E_x M_0 = E_x M_t = v(t,x). \qquad \square$$

Now that (1.3) has told us what the solution must be, the next logical step is to find conditions under which v is a solution. It is (and always will be) easy to show that if v is smooth enough, then it is a classical solution.

Theorem 1.4: Suppose that f is bounded. If $v \in C^{1,2}$ then it satisfies (1.1a).

Proof. The Markov property implies (see Exercise 2.1 in Chapter 1), that

$$E_x(f(B_t)|\mathcal{F}_s) = E_{B_s}(f(B_{t-s})) = v(t-s, B_s).$$

The left side is a martingale, so the right side is also a martingale. If $v \in C^{1,2}$, the by repeating the calculation in the proof of (1.2) shows that

$$\begin{aligned} v(t-s, B_s) - v(t, B_0) \ &= \ \int_0^s (-v_t + \frac{1}{2}\Delta v)(t-r, B_r)dr \\ &\qquad + \text{a local martingale.} \end{aligned}$$

The left side is a local martingale, so the integral on the right side is also a local martingale. But the integral is continuous and locally of bounded variation, so by (3.3) in chapter 2 it must be $\equiv 0$ almost surely. Since v_t and Δv are continuous, it follows that $-v_t + (1/2)\Delta v \equiv 0$. For if it were $\neq$ at some point (t, x), then it would be $\neq 0$ on an open neighborhood of that point, and, hence, with positive probability the integral would not be $\equiv 0$, a contradiction. $\square$

It is easy to give conditions that imply that v satisfies (1.1b). In order to keep the exposition simple, we first consider the situation when f is bounded.

Theorem (1.5): If f is bounded and continuous, then v satisfies (1.1b).

Proof: Note that $(B_t - B_0) \overset{d}{=} t^{1/2} Z$ where $Z \sim N(0, 1)$, so if $t_n \to 0$ and $x_n \to x$, the bounded convergence implies that

$$v(t_n, x) = Ef(x_n + t_n^{1/2} Z) \to f(x) \qquad \square$$

The final step in showing that v is a solution is to find conditions that guarantee that it is smooth. In the present case, this is relatively easy, as we now show.

Theorem (1.6): If f is bounded, then $v \in C^{1,2}$ and hence satisfies (1.1a).

Proof: By definition,

$$v(t,x) = E_x f(B_t) = \int f(y) p_t(x,y) dy$$

where $p_t(x,y) = (2\pi t)^{-d/2} e^{-|x-y|^2/2t}$. Writing $D_i = \partial/\partial x_i$ and $D_t = \partial/\partial t$, a little calculus yields:

$$\begin{aligned} D_i p_t(x,y) &= -\frac{(x_i - y_i)}{t} p_t(x,y) \\ D_{ii} p_t(x,y) &= \frac{(x_i - y_i)^2 - t}{t^2} p_t(x,y) \\ D_{ij} p_t(x,y) &= \frac{(x_i - y_i)(x_j - y_j)}{t^2} p_t(x,y) \qquad i \neq j \\ D_t p_t(x,y) &= \left(-\frac{d/2}{t} + \frac{|x-y|^2}{2t^2} \right) p_t(x,y). \end{aligned}$$

If f is bounded, then it is easy to see that for $\alpha = i,\ ij,$ or t,

$$\int |D_\alpha p_t(x,y) f(y)| dy < \infty$$

and is continuous in $\mathbb{R}^d$, so (1.6) follows from the following lemma on differentiating under the integral sign.

Lemma (1.7): Let $(S, \mathcal{S}, m)$ be a $\sigma-$finite measure space, and let $g : S \mapsto \mathbb{R}$ be measurable. Suppose that for $x \in G$, an open subset of $\mathbb{R}^d$ and some $h_0 > 0$ we have:
(a) $u(x) = \int_S K(x,y) g(y) dm(y)$ where K and $\partial K/\partial x_i : G \times S \mapsto \mathbb{R}$ are measurable functions with:
(b) $K(x^* + he_i, y) - K(x^*, y) = \int_0^h \frac{\partial K}{\partial x_i}(x^* + \theta e_i, y) d\theta$ for $|h| \leq h_0$ and $y \in S$.
(c) $u_i(x) = \int_S \frac{\partial K}{\partial x_i}(x,y) g(y) dm(y)$ is continuous at x^*.
and
(d) $\int_S \int_{-h_0}^{h_0} \left| \frac{\partial K}{\partial x_i}(x^* + \theta e_i, y) g(y) \right| d\theta dm(y) < \infty$.
Then $\partial u/\partial x_i$ exists at x^* and equals $u_i(x^*)$.

Proof: Using the definition of u in (a), then (b) and Fubini's theorem, which is justified for $|h| \le h_0$ by (d) we have

$$\begin{aligned} u(x^* + he_i) - u(x^*) &= \int_S (K(x^* + he_i, y) - K(x^*, y))g(y)dm(y) \\ &= \int_0^t \int_S \frac{\partial K}{\partial x_i}(x^* + \theta e_i, y)g(y)dm(y)d\theta. \end{aligned}$$

Dividing by h and letting $h \to 0$ the claimed result follows from (c). $\square$

We will also need a result about differentiating sums. Taking $S = \mathbf{Z}$ with $\mathcal{S} =$ all subsets of S, and μ to be counting measure in (1.7), then setting $g \equiv 1$ and $f_n(x) = K(x, n)$ yields the following. Note that in (a) and (c) of (1.7) it is implicit that the integrals exist, so here in (a) and (c) we assume that the sums converge absolutely:

Lemma 1.8 Suppose that for $x \in G$, an open subset of $\mathbb{R}^d$ and some $h_0 > 0$ we have:
(a) $u(x) = \sum_n f_n(x)$ where f_n and $\partial f_n/\partial x_i : G \mapsto \mathbb{R}$, $n \in \mathbf{Z}$, are measurable functions with:
(b) $f_n(x^* + he_i) - f_n(x^*) = \int_0^h \frac{\partial f_n}{\partial x_i}(x^* + \theta e_i) d\theta$ for $|h| \le h_0$ and $n \in \mathbf{Z}$.
(c) $u_i(x) = \sum_n \frac{\partial f_n}{\partial x_i}(x)$ is continuous at x^*,
and
(d) $\sum_n \int_{-h_0}^{h_0} \left| \frac{\partial f_n}{\partial x_i}(x^* + \theta e_i) \right| d\theta < \infty$.
Then $\partial u/\partial x_i$ exists at x^* and equals $u_i(x^*)$.

Unbounded f: For some applications, the assumption that f is bounded is too restrictive. To see what type of unbounded f we can allow, note that, at the bare minimum we need $E_x|f(B_t)| < \infty$ for all t. Since

$$E_x|f(B_t)| = \int \frac{1}{(2\pi t)^{d/2}} e^{-|y-x|^2/2t} |f(y)| dy,$$

a condition that guarantees this for locally bounded f is

$$x^{-2}\log^{+}|f(x)| \to 0 \quad \text{as} \quad x \to \infty. \qquad (*)$$

(Possibly relate this to an Orlicz norm condition? What condition on f is necessary and sufficient?)

Replacing the bounded convergence theorem in (1.5) and (1.6) by the dominated convergence theorem, it is not hard to prove the following:

Theorem (1.9): If f is continuous and satisfies $(*)$, then v satisfies (1.1).

4.2: The inhomogeneous equation

Now we consider what happens when we add a function $g(t, x)$ to the equation we considered in the last section; i.e we now study

(2.1a) $u \in C^{1,2}$ and $u_t = \frac{1}{2}\Delta u + g \quad$ in $(0, \infty) \times \mathbb{R}^d$.
(2.1b) u is continuous at each point of $\{0\} \times \mathbb{R}^d$ and $u(0, x) = f(x)$.

We observed in Section 4.1 and (2.1b) cannot hold unless f is continuous. Here $g = u_t - (1/2)\Delta u$ so the equation in (2.1a) cannot hold with $u \in C^{1,2}$ unless $g(t, x)$ is continuous.

The first step in treating the new equation is to observe that if u_1 is a solution of the equation with $f = f_0$ and $g = 0$ which we studied in the last section, and u_2 is a solution of the equation with $f = 0$ and $g = g_0$, then $u_1 + u_2$ is a solution of the equation with $f = f_0$ and $g = g_0$, so we can restrict our attention to the case $f \equiv 0$.

Having made this simplification, we will now study the equation above by following the procedure used in the last section. The first step is to find an associated local martingale.

Theorem 2.2: If u satisfies (2.1a), then

$$M_s = u(t-s, B_s) + \int_0^s g(t-r, B_r)dr$$

is a local martingale on $[0, t)$.

Proof: Applying Itô's formula as in the proof of (1.2) yields

$$\begin{aligned} u(t-s, B_s) - u(t, B_0) \; &= \; \int_0^s (-u_t + \frac{1}{2}\Delta u)(t-r, B_r)dr \\ &\quad + \; \int_0^s \nabla u(t-r, B_r) \cdot dB_r \end{aligned}$$

which proves (2.2), since $-u_t + \frac{1}{2}\Delta u = -g$ and the second term on the right side is a local martingale. □

The next step is, again, a uniqueness result.

Theorem 2.3: Suppose that g is bounded. If there is a solution of (2.1) that is bounded on $[0, T] \times \mathbb{R}^d$ for any $T < \infty$, it must be

$$v(t, x) \equiv E_x \left(\int_0^t g(t - s, B_s) ds \right).$$

Proof: Under the assumptions on g and u, M_s, $0 \leq s < t$, defined in (2.2) is a bounded martingale and $u(0, x) \equiv 0$, so

$$M_t \equiv \lim_{s \nearrow t} M_s = \int_0^t g(t - s) B_s ds$$

and since M_s is uniformly integrable,

$$u(t, x) = E_x M_0 = E_x M_t = v(t, x). \qquad \square$$

Again, it is relatively easy to show that if v is smooth enough it is a solution.

Theorem 2.4: Suppose that g is bounded and continuous. If $v \in C^{1,2}$, then it satisfies (2.1a) in $(0, \infty) \times \mathbb{R}^d$

Proof: Using the Markov property (see Exercise 2.6 in Chapter 1) gives

$$\begin{aligned}
&E_x\left(\int_0^t g(t-r,B_r)dr\middle|\mathcal{F}_s\right)\\
&= \int_0^s g(t-r,B_r)dr + E_{B_s}\left(\int_0^{t-s} g(t-s-u,B_u)du\right)\\
&= \int_0^s g(t-r,B_r)dr + v(t-s,B_s).
\end{aligned}$$

The left side is a martingale, so the right side is also a martingale. If $v \in C^{1,2}$, then repeating the calculation in the proof of (2.2) shows that

$$\begin{aligned}
&v(t-s,B_s) - v(t,B_0) + \int_0^t g(t-r,B_r)dr\\
&= \int_0^s (-v_t + \frac{1}{2}\Delta v + g)(t-r,B_r)dr\\
&\qquad + \text{a local martingale.}
\end{aligned}$$

The left side is a local martingale, so the integral on the right side is also a local martingale. Since the integral is continuous and locally of bounded variation, (3.3) in Chapter 2 implies it must be $\equiv 0$ almost surely. Again this implies that $(-v_t + \frac{1}{2}\Delta v + g) \equiv 0$, since our assumptions imply that this quantity is continuous, and if it were $\neq 0$ at some point (t, x), then we would have a contradiction. $\square$

Theorem 2.5: If g is bounded, then v satisfies (2.1b).

Proof: If $|g| \leq M$, then as $t \to 0$,

$$|v(t,x)| \leq E_x \int_0^t |g(t-s, B_s)| ds \leq Mt \to 0. \qquad \square$$

The final step in showing that v is a solution is to check that $v \in C^{2,1}$. Since the calculations needed to show these properties are quite tedious, we will content ourselves with statements of the results and indications of their proofs. In any case the upshot is:

Take home message: It is not enough to assume g is continuous to have $v \in C^{1,2}$. We must assume g to be **Hölder continuous locally in** t; that is, for any $N < \infty$ there are constants $C, \alpha \in (0, \infty)$ which may depend on N, such that $|g(t,x) - g(t,y)| \leq C|x-y|^{\alpha}$ whenever $t \leq N$.

The reason for this assumption can be found in the proof of (2.6c) and (2.6d) below.

The first step in showing $v \in C^{1,2}$ is to assume g is bounded and use Fubini's theorem to conclude

$$v(t,x) = \int_0^t \int_{\mathbb{R}^d} g(t-s,y) p_s(x,y) dy ds$$

where $p_s(x,y) = (2\pi s)^{-d/2} e^{-|y-x|^2/2s}$. The expression we have just written for v in the last display is what Friedman (1964) would call a volume potential and would write as

$$V(x,t) = \int_{T_0}^t \int_D Z(x,t;\xi,\tau) g(\xi,\tau) d\xi d\tau.$$

To translate between notations, set $T_0 = 0$, $D = \mathbb{R}^d$,

$$Z(x, t; \xi, \tau) = p_{t-\tau}(x, \xi),$$

and change variables $s = t-\tau$, $y = \xi$. Because of their importance for the parametrix method, the differentiability properties of volume potentials are well-known. The results we will state are just Theorems 2 and 5 in Chapter 1 of Friedman (1964), so the reader interested in knowing the whole story can find the missing details there.

Theorem 2.6a): If g is a bounded measurable function, then $v(t, x)$ is continuous on $(0, \infty) \times \mathbb{R}^d$.

Proof: Use the bounded convergence theorem. □

Theorem 2.6b): There is a constant C so that if $|g| \le M$ is measurable, then the partial derivatives $D_i v = \partial v / \partial x_i$ have $|D_i v| \le CMt^{1/2}$, are continuous, and are given by

$$D_i v = \int_0^t \int_{\mathbb{R}^d} D_i p_s(x, y) g(t-s, y) dy ds$$

Proof: Using the formula for $D_i p_s$ from (1.6), the right-side is

$$-\int_0^t \int_{\mathbb{R}^d} (2\pi s)^{-d/2} \frac{(x_i - y_i)}{s} e^{-|y-x|^2/2s} g(t-s, y) dy ds$$
$$= - \int_0^t \frac{ds}{s} E_x\{(x_i - B_s^i) g(t, s, B_s)\}.$$

Although the last formula looks suspicious because we are integrating s^{-1} near 0, everything is really all right: if $|g| \leq M$, then

$$E_x|(x_i - B_s^i) g(t, s, B_s)| \leq M E_x |x_i - B_s^i| = C M s^{1/2},$$

so we have

$$\int_0^t \frac{ds}{s} M E_x |x_i - B_s^i| \leq 2 C M t^{1/2} < \infty.$$

Using our result on differentiating under the integral sign, (1.7), it follows that the partial derivatives $D_i v$ exist, are continuous, and have the indicated form. □

It gets somewhat worse when we tackle the second derivatives:

Theorem 2.6c: Suppose that g is bounded and Hölder continuous locally in t. Then the partial derivatives $D_{ij}v = \partial^2 v/\partial x_i \partial x_j$ are continuous, and

$$D_{i,j}v = \int_0^t \int_{\mathbb{R}^d} D_{ij}p_s(x,y)g(t-s,y)dyds.$$

Proof: Suppose for simplicity that $i = j$. Using (1.6) giving the formula for $D_{i,i}$, the right side is

$$\int_0^t \int_{\mathbb{R}^d} (2\pi s)^{-d/2} \left(\frac{(x_i - y_i)^2 - s}{s^2} \right) e^{-|y-x|^2/2s} g(t-s,y)dyds$$
$$= \int_0^t E_x \left\{ \left(\frac{(x_i - B_s^i) - s}{s^2} \right) g(t-s,B_s) \right\} ds.$$

However, this time $E_x|(x_i - B_s^i)^2 - s| = sE_0|(B_1^i)^2 - 1|$, so

$$\int_0^t E_x \left| \frac{(x_i - B_s^i)^2 - s}{s^2} \right| ds = \infty.$$

However, if g is Hölder continuous at x, we can overcome this problem: the fact that $E_x(x_i - B_s^i)^2 = s$ allows us to write

$$\begin{aligned} & E_x\left[\left(\frac{(x_i - B_s^i)^2 - s}{s^2}\right) g(t-s, B_s)\right] \\ = \ & E_x\left[\left(\frac{(x_i - B_s^i)^2 - s}{s^2}\right) \{g(t-s, B_s) - g(t-s, x)\}\right]. \end{aligned}$$

By using the Hölder continuity of g one an show that the quantity in the last display is $\le Cs^{-1+\alpha/2}$, so its integral from $s = 0$ to t converges absolutely, and with a little work (2.6c) follows. (See Friedman (1964), pages 10 -12, for more details.)

The last detail concerns the derivative of $v(t, x)$ with respect to t:

Theorem 2.6d: Let g be as in (2.6c). Then $\partial v/\partial t$ exists, and

$$\frac{\partial v}{\partial t}(t, x) = g(t, x) + \int_0^t \int_{\mathbb{R}^d} \frac{\partial}{\partial t} p_{t-r}(x, y) g(r, y) dy.$$

4.3: The Feynman - Kac formula

Now we consider what happens when we add cu to the right side of the heat equation. That is, we will study:

(3.1a) $u \in C^{1,2}$ and $u_t = \frac{1}{2}\Delta u + cu$ in $(0, \infty) \times \mathbb{R}^d$.
(3.1b) u is continuous at each point of $\{0\} \times \mathbb{R}^d$ and
$u(0, x) = f(x)$.

If $c(t, x) \leq 0$, then this equation describes heat flow with cooling. That is, $u(t, x)$ gives the temperature at the point $x \in \mathbb{R}^d$ at time t, when the heat at x at time t dissipates at the rate $-c(t, x)$.

The first step is, as usual, to find a local martingale.

Theorem 3.2: Let $c_s^t = \int_0^s c(t - r, B_r) dr$. If u satisfies (3.1a), then

$$M_s = u(t - s, B_s) \exp(c_s^t)$$

is a local martingale on $[0, t)$.

Proof: Applying Itô's formula with $X_s^0 = t - s$, $X_s^i = B_s^i$ for $1 \le i \le d$, and $X_s^{d+1} = c_s^t$ yields

$$\begin{aligned} & u(t-s, B_s)\mathrm{exp}(c_s^t) - u(t, B_0) \\ & \quad = \int_0^s -u_t(t-r, B_r)\mathrm{exp}(c_r^t)dr + \int_0^s \mathrm{exp}(c_r^t)\nabla u(t-r, B_r) \cdot dB(r) \\ & \qquad + \int_0^s u(t-r), B_r)\mathrm{exp}(c_r^t)dc_r^t + \frac{1}{2}\int_0^s \Delta u(t-r, B_r)\mathrm{exp}(c_r^t)dr \end{aligned}$$

since we have

$$\langle X^i, X^j \rangle_t = \begin{cases} t & \text{if } 1 \le i = j \le d \\ 0 & \text{otherwise.} \end{cases}$$

Using $dc_r^t = c(t-r, B_r)dr$ and rearranging, the right side becomes

$$= \int \left(-u_t + cu + \frac{1}{2}\Delta u\right)(t-r, B_r)\mathrm{exp}(c_r^t)dr$$
$$+ \int_0^s \mathrm{exp}(c_r^t)\nabla u(t-r, B_r)\cdot dB(r),$$

which proves (3.2), since $-u_t + cu + \frac{1}{2}\Delta u = 0$ and the second term is a local martingale. $\square$

The next step is, once again, a uniqueness result.

Theorem 3.3: Suppose that c is bounded. If there is a solution of (3.1) that is bounded on $[0,T]\times\mathbb{R}^d$ for any $T<\infty$, it must be

$$v(t,x) \equiv E_x\{f(B_t)\mathrm{exp}(c_t^t)\}.$$

Proof: Under our assumptions on c and u, M_s, $0 \le s < t$, is a bounded martingale and $M_t = f(B_t)\mathrm{exp}(c_t^t)$. Since M_s is uniformly integrable it follows that

$$u(t,x) = E_xM_0 = E_xM_t = v(t,x). \qquad \square$$

As before, it is relatively easy to show that if v is smooth enough it is a solution.

Theorem 3.4: Suppose f is bounded and that c is bounded and continuous. If $v \in C^{1,2}$, then it satisfies (3.1a).

Proof: The Markov property implies, see Exercise 2.7 in Chapter 1 and take $h(r,x) = c(t-r,x)$, that

$$\begin{aligned} E_x(f(B_t)\text{exp}(c_t^t)|\mathcal{F}_s) &= \text{exp}(c_s^t)E_{B_s}(f(B_{t-s})\text{exp}(c_{t,s}^{t-s})) \\ &= \text{exp}(c_s^t)v(t-s,B_s). \end{aligned}$$

The left side is a martingale, so the right side is also a martingale. If $v \in C^{2,1}$, then repeating the calculation in the proof of (3.2) shows that

$$\begin{aligned} &v(t-s,B_s)\text{exp}(c_s^t) - v(t,B_0) \\ &\quad = \int_0^t (-v_t + cv + \frac{1}{2}\Delta v)(t-r,B_r)\text{exp}(c_r^t)dr \\ &\qquad + \text{a local martingale.} \end{aligned}$$

The left side is a local martingale, so the integral on the right side is also a local martingale. Since the integral is continuous and locally of bounded variation, (3.3) in Chapter 2 implies that it must be $\equiv 0$ almost surely. Again this implies that $(-v_t + cv + \frac{1}{2}\Delta v)(t-r, B_r)\exp(c_t^t) \equiv 0$ for our assumptions imply this quantity is continuous and if it were $\neq 0$ as some point we would have a contradiction. □

The next step is to give a condition that guarantees that v satisfies (3.1b). As before, we first consider the case in which everything is bounded.

Theorem 3.5: If c is bounded and f is bounded and continuous, then v satisfies (3.1b).

Proof: If $|c| \leq M$, then $e^{-Mt} \leq \exp(c_t^t) \leq e^{Mt}$, so $\exp(c_t^t) \to 1$ as $t \to 0$. This implies that

$$|E_x \exp(c_t^t) f(B_t) - E_x f(B_t)| \leq \|f\|_\infty E_x |\exp(c_t^t) - 1| \to 0.$$

(1.5) implies that $(t, x) \to E_x f(B_t)$ is continuous at each point of $\{0\} \times \mathbb{R}^d$, and the claim follows. $\square$

This brings us to the problem of determining when v is smooth enough to be a solution.

Theorem 3.6: Suppose that f is bounded and Hölder continuous. If c is bounded and Hölder continuous locally in t, then $v \in C^{1,2}$ and, hence, satisfies (3.1a).

Proof: To solve the problem in this case, we proceed by reducing the current case to the previous case. First note that

$$c_s^t = \int_0^s c(t-r, B_r)dr$$

is continuous and locally of bounded variation. So Itô's formula, (10.2) in Chapter 2, implies that if $h \in C^1$, then

$$h(c_t^t) - h(c_0^t) = \int_0^t h'(c_s^t)dc_s^t.$$

Taking $h(x) = e^{-x}$ we have

$$\text{exp}(-c^t_t) - 1 = -\int_0^t \text{exp}(-c^t_s)c(t-s, B_s)ds.$$

Multiplying by $-\text{exp}(c^t_t)$ gives

$$\text{exp}(c^t_t) - 1 = \int_0^t c(t-s, B_s)\text{exp}(c^t_t - c^t_s)ds.$$

Substitution of the definitions of c^t_t and c^t_s we have

$$\text{exp}\left(\int_0^t c(t-r, B_r)dr\right) = 1 + \int_0^t c(t-s, B_s)\text{exp}\left(\int_s^t c(t-r, B_r)dr\right)ds.$$

Multiplying by $f(B_t)$, taking expected values, and using Fubini's theorem, which is justified since everything is bounded, gives

$$v(t,x) \;=\; E_x f(B_t) + \int_0^t E_x\left\{c(t-s, B_s)\text{exp}\left(\int_s^t c(t-r, B_r)dr\right)f(B_t)\right\}ds.$$

Conditioning on $\mathcal{F}_s$, noticing that $c(t-s, B_s) \in \mathcal{F}_s$, and using the Markov property as in Exercise 2.7 of Chapter 1,

$$E_x\left(c(t-s,B_s)\text{exp}\left(\int_s^t c(t-r,B_r)dr\right)f(B_t)\middle|\mathcal{F}_s\right) = c(t-s,B_s)v(t-s,B_s).$$

Taking the expected value of the last equation and plugging it into the previous one yields

$$\begin{aligned} v(t,x) &= E_x f(B_t) + \int_0^t E_x\{c(t-s,B_s)v(t-s,B_s)ds \\ &\equiv v_1(t,x) + v_2(t,x). \qquad\qquad (*) \end{aligned}$$

The first term on the right, $v_1(t,x)$, is $C^{1,2}$ by (1.6). The second term, $v_2(t,x)$, is of the form considered in the previous section with $g(r,x) = c(r,x)v(r,x)$.

4.9: Laplace Transforms & Arcsine Laws

Example 9.3: Kac's (1951) derivation of Lévy's arcsine law. Let $H_t = \int_0^t 1_{[0,\infty)}(B_s)ds$ for $t > 0$. If $\theta \in [0,1]$, then

$$P_0(H_t \le \theta t) = \frac{1}{\pi}\int_0^\theta \frac{1}{\sqrt{r(1-r)}}dr = \frac{2}{\pi}\text{arcsin}(\sqrt{\theta}).$$

Remark: The reader should note that by scaling the distribution of H_t/t does not depend on t.

To prove the claimed formula, we begin with the following lemma:

Lemma 9.3: Let $c(x) = -\alpha - \beta 1_{[0,\infty)}(x)$ with α, $\beta \ge 0$. Suppose that v is bounded, C^1, and satisfies

$$\frac{1}{2}\Delta v + cv = -1$$

for all $x \neq 0$. Then

$$v(x) = -\int_0^\infty e^{-\alpha t} E_x e^{-\beta H_t} dt.$$

Proof: Our assumptions about v imply that

$$v''(x) = -2(1 + c(x)v(x))$$

in the sense of distribution. So two results from Chapter 2, the Meyer-Tanaka formula, (11.4) and (11.7) imply

$$v(B_t) - v(B_0) \;=\; \int_0^t v'(B_s)dB_s - \int_0^t (1 + c(B_s)v(B_s))ds.$$

Letting $c_t = \int_0^t c(B_s)ds$ and using the integration by parts formula (10.1) in Chapter 2, with $X_t = v(B_t)$ and $Y_t = \exp(c_t)$, which is locally of bounded variation, we have

$$\begin{aligned} v(B_t)\text{exp}(c_t) - v(B_0) \;=\; & \int_0^t \text{exp}(c_s)v'(B_s)dB_s \\ & - \int_0^t \text{exp}(c_s)\{1 + c(B_s)v(B_s)\}ds \\ & + \int_0^t v(B_s)\text{exp}(c_s)dc_s. \end{aligned}$$

Thus we find that $M_t = v(B_t)\text{exp}(c_t) + \int_0^t \text{exp}(c_s)ds$ is a local martingale. Since v is bounded and $c_t \leq 0$, M_t is bounded. As $t \to \infty$, $\text{exp}(c_t) \leq e^{-\alpha t} \to 0$, so, using the martingale and bounded convergence theorems gives

$$v(x) = E_x M_0 = E_x M_\infty = E_x \int_0^\infty \text{exp}(c_s)ds.$$

Plugging in the definition of $c(x)$ and using Fubini's theorem leads to the formula given above. □

(9.3) tells us that we want to find a bounded C^1 function v with

$$\begin{aligned}(\alpha+\beta)v &= \frac{1}{2}v'' + 1, \quad x > 0\\ \alpha v &= \frac{1}{2}v'' + 1, \quad x < 0.\end{aligned}$$

To solve the equation $\gamma v = \frac{1}{2}v'' + 1$ we write $v = v_0 + v_1$ where $v_1(x) \equiv 1/\gamma$ and note that

$$\begin{aligned}\gamma v_1 &= \frac{1}{2}v_1'' + 1\\ \gamma v_0 &= \frac{1}{2}v_0'',\end{aligned}$$

so we have $v_0(x) = C\exp(\pm x\sqrt{2\gamma})$. Choosing the signs to keep v bounded yields

$$v(x) = \begin{cases} Ae^{-x\sqrt{2(\alpha+\beta)}} + \frac{1}{\alpha+\beta} & x > 0\\ Be^{x\sqrt{2\alpha}} + \frac{1}{\alpha} & x < 0\end{cases}$$

To find A and B we note that we want $v \in C^1$, so this yields

$$\begin{aligned} A + \frac{1}{\alpha+\beta} \;&=\; B + \frac{1}{\alpha}, \\ -A\sqrt{2(\alpha+\beta)} &= B\sqrt{2\alpha}. \end{aligned}$$

Solving these equantions for A and B yields

$$A = \frac{\sqrt{\alpha+\beta}}{(\alpha+\beta)\sqrt{\alpha}}, \qquad B = -A\sqrt{\alpha+\beta}.$$

The quantity of interest for our problem is

$$v(0) = A + \frac{1}{\alpha+\beta} = \frac{1}{\sqrt{\alpha(\alpha+\beta)}} \; .$$

Now we will work backwards to the promised result. First note the identity

$$\int_0^\infty \frac{e^{-\gamma t}}{\sqrt{t}} dt \;=\; \int_0^\infty \sqrt{2/\gamma} e^{-x^2/2} dx = \sqrt{\frac{2}{\gamma}} \cdot \frac{1}{2}\sqrt{2\pi} = \sqrt{\pi/\gamma}.$$

This implies

$$\begin{aligned}
&\int_0^\infty e^{-\alpha t}\frac{1}{\pi}\int_0^t \frac{e^{-\beta s}}{\sqrt{s(t-s)}}dsdt \\
&= \frac{1}{\pi}\int_{s=0}^\infty \int_{t=0}^\infty 1_{[s\le t]}\frac{e^{-(\alpha t+\beta s)}}{\sqrt{s(t-s)}}dtds \\
&= \frac{1}{\pi}\int_{s=0}^\infty \int_{t=0}^\infty 1_{[s\le t]}\frac{e^{-(\alpha+\beta)s}}{\sqrt{s}}\cdot\frac{e^{-\alpha(t-s)}}{\sqrt{t-s}}dtds \\
&= \frac{1}{\pi}\int_{s=0}^\infty \frac{e^{-(\alpha+\beta)s}}{\sqrt{s}}\left(\int_{t=0}^\infty 1_{[s\le t]}\frac{e^{-\alpha(t-s)}}{\sqrt{t-s}}dt\right)ds \\
&= \frac{1}{\pi}\int_{s=0}^\infty \frac{e^{-(\alpha+\beta)s}}{\sqrt{s}}\left(\int_s^\infty \frac{e^{-\alpha(t-s)}}{\sqrt{t-s}}dt\right)ds \\
&= \frac{1}{\pi}\int_{s=0}^\infty \frac{e^{-(\alpha+\beta)s}}{\sqrt{s}}\left(\int_0^\infty \frac{e^{-\alpha r}}{\sqrt{r}}dr\right)ds = \frac{1}{\sqrt{(\alpha+\beta)\alpha}}\ .
\end{aligned}$$

By the uniqueness of Laplace transforms it follows that

$$E_0\text{exp}(-\beta H_t) = \frac{1}{\pi}\int_0^t \frac{e^{-\beta s}}{\sqrt{s(t-s)}}ds.$$

By uniqueness of Laplace transforms again we concluded that H_t has density

$$f_t(s) = \frac{1}{\pi}\frac{1}{\sqrt{s(t-s)}}1_{(0,t)}(s),$$

and hence $H_1 \sim$ Beta$(1/2, 1/2)$, with distribution function $F_1(u) = (2/\Pi)arcsin(\sqrt{u})$