# Math/Stat 523, Spring 2020

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Lecture 15

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# Outline, Brownian motion IV

- 1: Change of Time, Lévy's theorem;
  Section 3.4
- 2: Burkholder - Davis - Gundy;
  Section 3.5
- 3: Martingales adapted to Brownian filtrations; Section 3.6

# 1. Change of Time, Lévy's theorem; Section 3.4

Here we will prove Lévy's characterization of Brownian motion (4.1), and use it to show that every continuous local martingale is a time change of Brownian motion.

**Theorem 4.1:** If $X_t$ is a continuous local martingale with $X_0 = 0$ and $\langle X \rangle_t \equiv t$, then $X_t$ is a one dimensional Brownian motion.

**Proof:** By (4.5) in Chapter 1 it suffices to prove the following Lemma:

**Lemma 4.2:** For any $s$ and $t$, $X_{s+t} - X_s$ is independent of $\mathcal{F}_s$ and has a normal distribution with mean 0 and variance $t$.

**Proof of (4.2):** Applying the complex version of Itô's formula, (7.9) in Chapter 2, to $Y_r = X_{s+r} - X_s$ and $f(x) = e^{i\theta x}$, we get

$$e^{i\theta Y_t} - 1 = i\theta \int_0^t e^{i\theta Y_u} dY_u - \frac{\theta^2}{2} \int_0^t e^{i\theta Y_u} du.$$

Let $\mathcal{G}_r \equiv \mathcal{F}_{s+r}$ and let $A \in \mathcal{F}_s = \mathcal{G}_0$.

The first term on the right, which we will call $Z_t$, is a local martingale w.r.t. $\mathcal{G}_t$. To get rid of that term, let $T_n \nearrow \infty$ be a sequence of stopping times that reduces $Z_t$, replace $t$ by $t \wedge T_n$, and integrate over $A$. The definition of conditional expectation implies

$$E(Z_{t\wedge T_n} 1_A) = E(E(Z_{t\wedge T_n}|\mathcal{G}_0)1_A) = E(Z_0 1_A) = 0$$

since $Z_0 = 0$. So we have

$$E(e^{i\theta Y_{t\wedge T_n}} 1_A) - P(A) = 0 - \frac{\theta^2}{2} E\left(\int_0^{t\wedge T_n} e^{i\theta Y_u} du \cdot 1_A\right).$$

Since $|e^{i\theta x}| = 1$, letting $n \to \infty$ and using the bounded convergence theorem gives

$$\begin{aligned} E(e^{i\theta Y_t} 1_A) - P(A) &= -\frac{\theta^2}{2} E\left(\int_0^t e^{i\theta Y_u} du \cdot 1_A\right) \\ &= -\frac{\theta^2}{2} \int_0^t E\left(e^{i\theta Y_u} \cdot 1_A\right) du \end{aligned}$$

by Fubini's theorem (since the integrand is bounded and the two measures are finite). Writing $j(t) = E(e^{i\theta Y_t} 1_A)$, the last equality says

$$j(t) - P(A) = -\frac{\theta^2}{2} \int_0^t j(u) du.$$

Since we know that $|j(s)| \leq 1$, it follows that $|j(t) - j(u)| \leq |t-u|\theta^2/2$, so $j$ is continuous and we can differentiate across the equality in the last display to conclude $j$ is differentiable with

$$j'(t) = -\frac{\theta^2}{2} j(t).$$

Together with $j(0) = P(A)$, this shows that $j(t) = P(A)e^{-\theta^2 t/2}$, or

$$E(e^{i\theta Y_t} 1_A) = e^{-\theta^2 t/2} E 1_A.$$

Since this holds for all $A \in \mathcal{G}_0$ it follows that

$$E(e^{i\theta Y_t}|\mathcal{G}_0) = e^{-\theta^2 t/2},$$

or, in words, the conditional characteristic function of $Y_t$ is that of the normal distribution with mean 0 and variance $t$.

To use this to prove (4.2), we first take expected values of both sides to conclude that $Y_t$ has a normal distribution with mean 0 and variance $t$. The fact that the conditional characteristic function is a constant suggests that $Y_t$ is independent of $\mathcal{G}_0$. To turn this intuition into a proof, let $g$ be a $C^1$ function with compact support, and let

$$\varphi(\theta) = \int e^{i\theta x} g(x) dx$$

be its Fourier transform. We have assumed more than enough to conclude that $\varphi$ is integrable and hence

$$g(x) = \frac{1}{2\pi} \int e^{i\theta x} \varphi(-\theta) d\theta$$

Multiplying both sides of (4.3) by $\varphi(-\theta)$ and integrating shows that $E(g(Y_t)|\mathcal{G}_0)$ is a constant and hence

$$E(g(Y_t)|\mathcal{G}_0) = Eg(Y_t).$$

A monotone class argument now shows that the last conclusion holds for any bounded measurable $g$ Taking $g = 1_B$ and integrating the last equality over $A \in \mathcal{G}_0$ we have

$$P(A)P(Y_t \in B) = \int_A E(1_B(Y_t)|\mathcal{G}_0)dP = P(A \cap [Y_t \in B])$$

by the definition of conditional expectation, so we have proved the claimed independence. $\square$

**Exercise 4.1.** Suppose that $X_t^i$, $1 \le i \le d$, are continuous local martingales with $X_0 = 0$ and

$$\langle X^i, X^j \rangle = \begin{cases} t, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

Then $X_t = (X_t^1, \ldots, X_t^d)$ is a $d-$dimensional Brownian motion.

An immediate consequence of (4.1) is:

**Theorem 4.4:** Every continuous local martingale with $X_0 = 0$ and having $\langle X \rangle_\infty \equiv \infty$ is a time change of Brownian motion: if we let $\gamma(u) \equiv \inf\{t : \ \langle X \rangle_t > u\}$, then $B_u = X_{\gamma(u)}$ is a Brownian motion and $X_t = B_{\langle X \rangle_t}$.

**Proof:** Since $\gamma(\langle X \rangle_t) = t$, the second equality is an immediate consequence of the first. To prove the first equality. Recall from Exercise 3.8 of chapter 2 that $u \mapsto B_u$ is continuous, so it suffices to show that $B_u$ and $B_u^2 - u$ are local martingales.

**Lemma 4.5:** $B_u$, $\mathcal{F}_{\gamma(u)}$ is a local martingale.

**Proof of (4.5):** Let $T_n = \inf\{t : |X_t| > n\}$. The optional stopping theorem implies that if $u < v$ then

$$E(X_{\gamma(v)\wedge T_n})|\mathcal{F}_{\gamma(u)}) = X_{\gamma(u)\wedge T_n}$$

where we have used Exercise 2.1 in Chapter 2 to replace $\mathcal{F}_{\gamma(u)\wedge T_n}$ by $\mathcal{F}_{\gamma(u)}$. To let $n \to \infty$ we note that the $L^2$maximal inequality, the fact that $X^2_{\gamma(v)\wedge T_n} - \langle X \rangle_{\gamma(v)\wedge T_n}$ is a martingale, and the definition of $\gamma(v)$ imply

$$\begin{aligned} E \sup_n X^2_{\gamma(v)\wedge T_n} &\leq 4 \sup_n E X^2_{\gamma(v)\wedge T_n} \\ &= 4 \sup_n E\langle X \rangle_{\gamma(v)\wedge T_n} \leq 4v \end{aligned}$$

The last result and the dominated convergence theorem imply that as $n \to \infty$, $X_{\gamma(v)\wedge T_n} \to X_{\gamma(t)}$ in $L^2$ for $t = u, v$. Since conditional expectation is a contraction in $L^2$, it follows that $E(X_{\gamma(v)\wedge T_n}|\mathcal{F}_{\gamma(u)}) \to_2 E(X_{\gamma(v)}|\mathcal{F}_{\gamma(u)})$ in $L^2$, and the proof is complete. □

To complete the proof of (4.4) it now remains to show:

**Lemma 4.6:** $B_u^2 - u$, $\mathcal{F}_{\gamma(v)}$ is a local martingale.

**Proof of (4.6)**: As in the proof of (4.5), the optional stopping theorem implies that if $u < v$ then

$$E(X^2_{\gamma(v)\wedge T_n} - \langle X \rangle_{\gamma(v)\wedge T_n} | \mathcal{F}_{\gamma(u)}) = X^2_{\gamma(u)\wedge T_n} - \langle X \rangle_{\gamma(u)\wedge T_n}.$$

To let $n \to \infty$ we observe that since $(a+b)^2 \le 2(a^2+b^2)$, (3.4) and the definition of $\gamma(u)$, then

$$\begin{aligned} E \sup_n \left( X^2_{\gamma(v)\wedge T_n} - \langle X \rangle_{\gamma(v)\wedge T_n} \right)^2 &\le 2E \sup_n X^4_{\gamma(v)\wedge T_n} + 2E\langle X \rangle^2_{\gamma(v)} \\ &\le CE\langle X \rangle^2_{\gamma(v)} \le Cv^2. \end{aligned}$$

The proof can now be completed as in (4.5) by using the dominated convergence theorem, and the fact that conditional expectation is a contraction in $L^2$. □

The next goal is to extend (4.4) to $X_t$ with $P(\langle X\rangle_\infty < \infty) > 0$. In this case $X_{\gamma(u)}$ is a Brownian motion run for an amount of time $\langle X\rangle_\infty$. The first step in nailing this down is the following lemma:

**Lemma 4.7:** $\lim_{t\nearrow\infty} X_t$ exists almost surely on $\{\langle X\rangle_\infty < \infty\}$.

**Proof:** Let $T_n \equiv \inf\{t : \langle X\rangle_t \geq n\}$. By (3.7) in Chapter 2 it follows that

$$\langle X^{T_n}\rangle_t = \langle X\rangle_{t\wedge T_n} \leq n.$$

Using this with Exercise 4.3 in Chapter 2 we get $X(t\wedge T_n) \in \mathcal{M}^2$, so $\lim_{t\to\infty} X_{t\wedge T_n}$ exists almost surely and in $L^2$. This shows that $\lim_{t\to\infty} X_t$ exists almost surely on $\{T_n = \infty\} \supset \{\langle X\rangle_\infty < n\}$. Letting $n \to \infty$ yields (4.7). □

To prove the promised extension of (4.4), now let $\gamma(u) = \inf\{t : \langle X\rangle)_t > u\}$ when $u < \langle X\rangle_\infty$, let $X_\infty = \lim_{\to\infty} X_t$ on $\{\langle X\rangle_\infty < \infty\}$, let $B_t$ be a Brownian motion which is independent of $\{X_t :\ t > 0\}$, and let

$$Y_u = \begin{cases} X_{\gamma(u)} & u < \langle X\rangle_\infty \\ X_\infty + B(u - \langle X\rangle_\infty) & u \geq \langle X\rangle_\infty. \end{cases}$$

**Theorem 4.8:** $Y$ is a Brownian motion.

**Proof:** By (4.1) it suffices to show that $Y_u$ and $Y_u^2 - u$ are local martingales with respect to the filtration $\sigma[Y_t :\ t \leq u\}$. This holds on $[0, \langle X\rangle_\infty]$ for reasons indicated in the proof of (4.4). It holds on $[\langle X\rangle_\infty, \infty)$ because $B$ is a Brownian motion independent of $X$. □

One reason for interest in (4.8) is because it leads to a converse of (4.7).

**Theorem 4.9:** The following sets are equal almost surely:

$$C = \{\lim_{t\to\infty} X_t \ \text{ exists}\} \quad B = \{\sup_t |X|_t < \infty\}$$
$$A = \{\langle X\rangle_\infty < \infty\} \quad B_+ = \{\sup_t X_t < \infty\}.$$

**Proof:** Clearly, $C \subset B \subset B_+$. In section 3.1 we showed that Brownian motion has $\limsup_{t\to\infty} = \infty$. This and (4.8) implies that $A^c \subset B_c^+$, or $B_+ \subset A$. Finally, (4.7) implies $A \subset C$. $\square$

The result in (4.8) can be used to justify the assertion we made at the beginning of Sections 2.6 that $\Pi_3(X)$ is the largest possible class of of integrands. Suppose $H \in \Pi$ and let $T = \sup\{t : \int_0^t H_s^2 d\langle X\rangle_s < \infty\}$. Now $(H\cdot X)_t$ can be defined for $t < T$ and has

$$\langle H\cdot X\rangle_t = \int_0^t H_s^2 d\langle X\rangle_s,$$

so on $\{\langle H\cdot X\rangle_T = \infty\} \supset \{T < \infty\}$ we have

$$\limsup_{t\nearrow T} (H\cdot X)_t = \infty, \qquad \liminf_{t\nearrow T} (H\cdot X)_t = -\infty$$

and there is no reasonable way to continue to define $(H \cdot X)_t$ for $t \geq T$.

Convergence is not the only property of local martingales that can be studied using (4.9). Almost any almost-sure property concerning the Brownian path can be translated into a corresponding result for local martingales. This immediately gives us a number of theorems about the behavior of paths of local martingales. One example of this is:

**Law of the Iterated Logarithm (4.10):** Let $L(t) = \sqrt{2t\text{loglog}t}$ for $t \geq e$. Then on $\{\langle X \rangle_\infty = \infty\}$,

$$\limsup_{t \to \infty} \frac{X_t}{L(\langle X \rangle_t)} = 1 \quad \text{a.s.}$$

**Proof:** This follows from (4.9) and the result for Brownian motion proved in Section 7.9 of Durrett (1995). □

**Exercise 4.2:** Suppose $h : [0, \infty) \to \mathbb{R}$ is measurable and locally bounded. Use (4.4) to generalize Exercise 6.7 in Chapter 2 and conclude that

$$X_t = \int_0^t h_s dB_s \ \sim \ N(0, \sigma_h^2(t)) \quad \text{with} \quad \sigma_h^2(t) \equiv \int_0^t h^2(s) ds.$$

# 2. Burkholder-Davis-Gundy inequalities; Section 3.5

Let $X_t$ be a local martingale with $X_0 = 0$ and let $X_t^* = \sup_{s \le t} |X_s|$. This section is devoted to a proof of the following inequalities.

**Theorem 5.1: Burkholder-Davis-Gundy** For any $0 < p < \infty$ there are constants $0 < c, C < \infty$ so that

$$cE\langle X\rangle_t^{p/2} \le E(X_t^*)^p \le CE\langle X\rangle_t^{p/2}$$

**Remark:** This pair of inequalities should be contrasted with the $L^p$ maximal inequality for martingales that holds only for $1 < p < \infty$:

$$E(X_t^*)^p \le \left(\frac{p}{p-1}\right)^p E|X_t|^p.$$

Lévy's theorem (4.4) tells use that any continuous local martingale is a time change of Brownian motion $B_t$, so it suffices to let $B_t^* = \sup_{s \le t} |B_s|$ and prove the following theorem:

**Theorem 5.2:** For any $0 < p < \infty$ there are constants $0 < c, C < \infty$ so that for any stopping time $\tau$

$$cE\tau^{p/2} \le E(B_\tau^*)^p \le CE\tau^{p/2}.$$

**Proof of (5.2):** The crux of the proof is the following pair of inequalities:

**Lemma (5.3):** Let $\beta > 1$ and $\delta > 0$. Then for any $\lambda > 0$:
(a) $P(B_\tau^* > \beta\lambda,\ \tau^{1/2} \le \delta\lambda) \le \frac{\delta^2}{(\beta-1)^2} P(B_\tau^* > \lambda)$.
(b) $P(\tau^{1/2} > \beta\lambda,\ B_\tau^* \le \delta\lambda) \le \frac{\delta^2}{\beta^2-1} P(\tau^{1/2} > \lambda)$.

# 3. Martingales adapted to Brownian Filtrations; Section 3.6

Let $\{\mathcal{B}_t, t \geq 0\}$ be the filtration generated by a $d-$dimensional Brownian motion $B_t$ with $B_0 = 0$, defined on some probability space $(\Omega, \mathcal{F}, P)$. In this section we will show:
(i) all local martingales adapted to $\{\mathcal{B}_t,\ t \geq 0\}$ are continuous; and
(ii) Every random variable $X \in L^2(\Omega, \mathcal{B}_\infty, P)$ can be written as a stochastic integral.

**Theorem (6.1):** All local martingales adapted to $\{\mathcal{B}_t,\ t \geq 0\}$ are continuous.

Let $B_t = (B_t^1, \ldots, B_t^d)$ with $B_0 = 0$ and let $\{\mathcal{B}_t^i, t \geq 0\}$ be the filtrations generated by the coordinates.

**Theorem (6.2):** For any $X \in L^2(\Omega, \mathcal{B}_\infty, P)$ there are unique $H^i \in \Pi_2(B^i)$ with

$$X = EX + \sum_{i=1}^{d} \int_0^\infty H_s^i dB_s^i \equiv EX + \int_0^\infty \underline{H}_s \cdot d\underline{B}_s.$$