# Math/Stat 523, Spring 2020

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Lecture 14

*Monday May 18, Wednesday May 20*

# Outline

- 1: Brownian motion, II: recurrence and transience; Section 3.1
- 2: Brownian motion, II: occupation times; Section 3.2

# 1. Brownian Motion II; Sections 3.1 - 3.4

Here our goal is to use Itô's formula to develop further understanding of BM and local martingales more generally.

**Section 3.1: Recurrence and Transience** Suppose that $B_t$ is a $d-$dimensional Brownian motion and $B_t^i$ is the $i$th component, then $B_t^i$ is a martingale with $\langle B^i \rangle_t = t$. If $i \neq j$ Exercise 2.2 in Chapter 1 tells us that $B_t^i B_t^j$ is a martingale, so (3.11) in Chapter 2 implies $\langle B^i, B^j \rangle = 0$. Using this information in Itô's formula we see that if $f : \mathbb{R}^d \to \mathbb{R}$ is $C^2$, then

$$f(B_t) - f(B_0) = \sum_{i=1}^{d} \int_0^t D_i f(B_s) dB_s^i + \frac{1}{2} \sum_{i=1}^{d} D_{ii} f(B_s) ds.$$

Writing $\nabla f = (D_1 f, \ldots, D_d f)$ for the gradient of $f$, and $\Delta f = \sum_{i=1}^d D_{ii} f$ for the Laplacian of $f$, we can write the last equation more neatly as

$$f(B_t) - f(B_0) = \int_0^t \nabla f(B_s) \cdot dB_s + \frac{1}{2} \int_0^t \Delta f(B_s) ds. \qquad (1.1)$$

Here the dot in the first term stands for the inner product of two vectors ... and the precise meaning of that is given in the previous equation.

Functions with $\Delta f = 0$ are called **harmonic**. (1.1) shows that if we compose a harmonic function with Brownian motion, the result is a local martingale. The next result (and judicious choices of harmonic functions) is the key to deriving properties of Brownian motion from Itô's formula.

**Theorem 1.2:** let $G$ be a bounded open set and let $\tau = \inf\{t > 0 : B_t \notin G\}$. If $f \in C^2$ and $\Delta f = 0$ in $G$, and $f$ is continuous on the closure of $G$, $\overline{G}$, then for $x \in G$ we have $f(x) = E_x f(B_\tau)$.

**Proof:** The first step is to show that $P_x(\tau < \infty) = 1$. Let $K = \sup\{|x-y| : \ x, y \in G\}$ be the diameter of $G$. If $x \in G$ and $|B_1 - x| > K$, then $B_1 \notin \overline{G}$ and $\tau < 1$. Thus

$$P_x(\tau < 1) \geq P_x(|B_1 - x| > K) = P_0(|B_1| > K) = \epsilon_K > 0.$$

This shows $\sup_x P_x(\tau \geq k) \leq (1 - \epsilon_K)^k$ holds when $k = 1$. To prove the last result by induction on $k$ we observe that if $p_t(s, y)$ is the transition probability for Brownian motion, the Markov property implies

$$\begin{aligned} P_x(\tau > k) &\leq \int_G p_1(x, y) P_y(\tau \geq k-1) dy \\ &\leq (1 - \epsilon_K)^{k-1} P_x(B_1 \in G) \leq (1 - \epsilon_K)^k \end{aligned}$$

where the last two inequalities follows from the induction assumption and the fact that $|B_1 - x| > K$ implies $B_1 \notin G$. The last result implies that $P_x(\tau < \infty) = 1$ for all $x \in G$, and moreover that

$$\sup_x E_x \tau^p < \infty \quad \text{for all} \quad 0 < p < \infty.$$

To get the last conclusion recall that (see e.g. (5.7) in Chapter 1 of Durrett (1995))

$$E_x \tau^p = \int_0^\infty p t^{p-1} P_x(\tau > t) dt.$$

When $\Delta f = 0$ in $G$, (1.1) implies that $f(B_t)$ is a local martingale on $[0, \tau)$. We have assumed that $f$ is continuous on $\overline{G}$ and $G$ is bounded, so $f$ is bounded and if we apply the time change $\gamma$ defined in Section 2.2, $X_t = f(B_{\gamma(t)})$ is a bounded martingale (with w.r.t. $\mathcal{G}_t = \mathcal{F}_{\gamma(t)}$). Being a bounded martingale, $X_t$ converges almost surely to a limit $X_\infty$ which has $X_t = E_x(X_\infty | \mathcal{G}_t)$ and hence $E_x X_t = E_x X_\infty$. Since $\tau < \infty$ and $f$ is continuous on $\overline{G}$, $X_\infty = f(B_\tau)$. Taking $t = 0$ it follows that $f(x) = E_x X_0 = E_x X_\infty = E_x f(B_\tau)$. □

Now we will use (1.2) to prove several results concerning the range of Brownian motion $\{B_t : \ t \geq 0\}$, beginning with the one-dimensional case:

**Theorem 1.4:** Let $a < x < b$ and $T \equiv \inf\{t > 0 : \ B_t \notin (a, b)\}$. Then

$$P_x(B_T = a) = \frac{b-x}{b-a}, \qquad P_x(B_T = b) = \frac{x-a}{b-a}.$$

**Proof:** $f(x) = (b-x)/(b-a)$ has $f'' = 0$ in $(a, b)$, is continuous on $[a, b]$, and $f(a) = 1$, $f(b) = 0$, so (1.2) implies $f(x) = E_x f(B_T) = P_x(B_T = a)$. $\square$

**Remark:** Note that this proof of Theorem 1.4 is different than the proof we used in Math/Stat 522 in Winter Quarter.

Now let $T_x = \inf\{t > 0 : \ B_t = x\}$. From (1.4), it follows immediately that:

**Theorem 1.5:** For all $x$ and $y$, $P_x(T_y < \infty) = 1$.

**Proof:** Since $P_x(T_y < \infty) = P_{x-y}(T_0 < \infty)$, it suffices to prove the result when $y = 0$. By a reflection argument it is easily seen that we can assume $x > 0$. Now by using (1.4) it follows that $P_x(T_0 < T_{Mx}) = (M-1)/M$, and the right side approaches 1 as $M \to \infty$. $\square$

It is easy to improve (1.5) to conclude the following

**Theorem 1.6:** For any $s < \infty$, $P_x(B_t = y \text{ for some } t \geq s) = 1$.

**Proof:** By the Markov property

$$P_x(B_t = y \text{ for some } t \geq s) = E_x\{P_{B(s)}(T_y < \infty)\} = 1.$$

Theorem 1.6 implies that for any $y$ with probability 1 there is a sequence of times $t_n \nearrow \infty$ (which will depend on the outcome $\omega$) so that $B_{t_n} = y$, a conclusion we will hereafter abbreviate to "$B_t = y$ infinitely often" or "$B_t = y$ i.o." In the terminology of Markov process theory, we have shown that one-dimensional Brownian motion is recurrent.

**Exercise 1.2:** Use (1.6) to deduce that

$$-\infty =_{a.s.} \liminf_{t\to\infty} B_t < \limsup_{t\to\infty} B_t =_{a.s.} +\infty.$$

(In fact, by the LIL for Brownian motion we know even more:

$$-1 =_{a.s.} \liminf_{t\to\infty} \frac{B_t}{\sqrt{2t \log\log t}} < \limsup_{t\to\infty} \frac{B_t}{\sqrt{2t \log\log t}} =_{a.s.} 1.)$$

To study Brownian motion in $d \geq 2$ dimensions, we need to find some appropriate harmonic functions. In view of the spherical symmetry of Brownian motion, an obvious way to do this is to let $\varphi(x) = f(|x|^2)$ and then choose $f : \mathbb{R} \to \mathbb{R}$ so that $\Delta\varphi = 0$. In fact we use $|x|^2 = x_1^2 + \cdots + x_d^2$ rather than $|x|$ because it is easier to differentiate:

$$\begin{aligned} D_i f(|x|^2) &= f'(|x|^2) 2x_i, \\ D_{ii} f(|x|^2) &= f''(|x|^2) 4x_i^2 + 2f'(|x|^2). \end{aligned}$$

Therefore, for $\Delta\varphi = 0$ we need to choose $f$ so that

$$\begin{aligned} 0 &= \sum_{i=1}^{d} \{f''(|x|^2) + 2df'(|x|^2)\} \\ &= 4|x|^2 f''(|x|^2) + 2df'(|x|^2). \end{aligned}$$

Letting $y = |x|^2$, we can write the last display as $4yf''(y) + 2df'(y) = 0$, or, if $y > 0$

$$f''(y) = -\frac{d}{2y} f'(y).$$

Taking $f'(y) = Cy^{-d/2}$ guarantees $\Delta\varphi = 0$ for $x \neq 0$, so by choosing $C$ appropriately we can take

$$\varphi(x) = \begin{cases} \log|x|, & d = 2 \\ |x|^{2-d}, & d \geq 3. \end{cases}$$

Now can make use of (1.2) for $d \geq 2$. Let $S_r = \inf\{t : \ |B_t| = r\}$, and let $R > r$. Since $\varphi$ satisfies $\Delta\varphi = 0$ in $G = \{x : \ r < |x| < R\}$, and is continuous on $\overline{G}$, (1.2) implies

$$\varphi(x) = E_x\varphi(B_\tau) = \varphi(r)P_x(S_r < S_R) + \varphi(R)(1 - P_x(S_r < S_R))$$

where $\varphi(r)$ is short for the value of $\varphi(x)$ on $\{x : |x| = r\}$. Solving now gives

$$P_x(S_r < S_R) = \frac{\varphi(R) - \varphi(x)}{\varphi(R) - \varphi(r)}. \qquad (1.7)$$

When $d = 2$, the last display says

$$P_x(S_r < S_R) = \frac{\log R - \log|x|}{\log R - \log r}. \qquad (1.8)$$

If we fix $r$ and let $R \to \infty$ in (1.8), the right side converges to 1. So

$$P_x(S_r < \infty) = 1 \quad \text{for any} \quad x \quad \text{and any } r > 0,$$

and repeating the proof of (1.6) proves the following theorem:

**Theorem (1.9):** Two-dimensional Brownian motion is **recurrent** in the sense that if $G$ is any open set, then $P_x(B_t \in G \; i.o.) = 1$.

If we fix $R$, let $r \to 0$ in (1.8), and let $S_0 = \inf\{t > 0 : \; B_t = 0\}$, then for $x \neq 0$

$$P_x(S_0 < S_R) \leq \lim_{r \to 0} P_x(S_r < S_R) = 0.$$

Since this holds for all $R$ and since the continuity of Brownian paths implies $S_R \nearrow \infty$ as $R \nearrow \infty$, we have $P_x(S_0 < \infty) = 0$ for all $x \neq 0$. To extend this to $x = 0$, note that by the Markov property

$$P_0(B_t = 0 \text{ for some } t \geq \epsilon) = E_0[P_{B_\epsilon}(T_0 < \infty)] = 0$$

for all $\epsilon > 0$, so $P_0(B_t = 0$ for some $t > 0) = 0$, and thanks to our definition of $S_0 = \inf\{t > 0 : \; B_t = 0\}$ we have

$$P_x(S_0 < \infty) = 0 \quad \text{for all } x.$$

Thus in $d \geq 2$ Brownian motion will not hit 0 at a positive time (even if it starts there).

**Exercise 1.3:** Use the continuity of the Brownian path and $P_x(S_0 = \infty) = 1$ to show that if $x \neq 0$, then $P_x(S_r \nearrow \infty \text{ as } r \searrow 0) = 1$.

When $d \geq 3$, (1.7) says

$$P_x(S_r < S_R) = \frac{R^{2-d} - |x|^{2-d}}{R^{2-d} - r^{2-d}}. \tag{1.11}$$

Now there is no point in fixing $R$ and letting $r \to 0$: the fact that two-dimensional BM does not hit points implies that 3-dimensional BM does not hit points, and indeed will not hit the line $\{x: \ x_1 = x_2 = 0\} = \{x \in \mathbb{R}^3: \ x = (0, 0, x_3), \ x_3 \in \mathbb{R}\}$. If we fix $r$ and let $R \to \infty$ in (1.11) we get

$$P_x(S_r < \infty) = (r/|x|)^{d-2} < 1 \quad \text{if} \ \ |x| > r. \qquad (1.12).$$

From the last display it follows that for $d \geq 3$, BM is **transient**; i.e. it does not return infinitely often to any bounded set.

**Theorem 1.13:** As $t \to \infty$, $|B_t| \to \infty$ a.s.

**Proof:** Let $A_n \equiv \{|B_t| > n^{1/2} \ \text{ for all } t \geq S_n\}$, and note that $S_n < \infty$ by (1.3). The strong Markov property implies

$$P_x(A_n^c) = E_x(P_{B(S_n)}(S_{n^{1/2}} < \infty)) = (n^{1/2}/n)^{d-2} \to 0.$$

But then $\limsup A_n = \cap_{N=1}^{\infty} \cup_{n=N}^{\infty} A_n$ has

$$P(\limsup A_n) \geq \limsup P(A_n) = 1.$$

That is, infinitely often the Brownian path in $\mathbb{R}^d$ with $d \geq 3$ never returns to $\{x: \ |x| \leq n^{1/2}\}$ after time $S_n$, and this implies the desired result.

Dvoretzky and Erdös (1951) have proved the following result about how fast Brownian motion goes to $\infty$ in $d \geq 3$:

**Theorem 1.14:** Suppose $g(t)$ is positive and decreasing. Then

$$P_0(|B_t| \leq g(t)\sqrt{t} \text{ i.o. as } t \nearrow \infty) = 1 \text{ or } 0$$

according as

$$\int^{\infty} t^{-1} g(t)^{d-2} dt \begin{cases} = \infty \\ < \infty. \end{cases}$$

since

$$\int^{\infty} t^{-1} (\log t)^{-\alpha} dt = \begin{cases} = \infty \\ < \infty \end{cases}$$

according as $\alpha \leq 1$ or $\alpha > 1$, so $|B_t|$ goes to $\infty$ faster than $\sqrt{t}/(\log t)^{\alpha/(d-2)}$ for any $\alpha > 1$.

# 2: Brownian motion, II: occupation times; Section 3.2

Now let $D = B(0,r) = \{y : \; |y| < r\}$, the ball of radius $r$ centered at 0. In Section 3.1 we learned that $B_t$ will return to $D$ i.o. in $d \le 2$, but not in $d \ge 3$. In this section we study the occupation time $\int_0^\infty 1_D(B_t)dt$ and show that for any $x$

$$P_x\left(\int_0^\infty 1_D(B_t)dt = \infty\right) = 1 \quad \text{in } d \le 2, \tag{2.1}$$

$$E_x \int_0^\infty 1_D(B_t)dt < \infty \quad \text{in } d \ge 3. \tag{2.2}$$

**Proof of (2.1)** Let $T_0 = 0$ and $G = B(0,2r)$. For $k \ge 1$, let

$$\begin{aligned} S_k &= \inf\{t > T_{k-1} : \; B_t \in D\}, \\ T_k &= \inf\{t > S_k : \; B_t \in G\}. \end{aligned}$$

Writing $\tau$ for $T_1$ and using the strong Markov property, we find for $k \ge 1$

$$P_x\left(\int_{S_k}^{T_k} 1_D(B_t)dt \ge s \,\middle|\, \mathcal{F}_{S_k}\right) = P_{B(S_k)}\left(\int_0^\tau 1_D(B_t) \ge s\right) = H(s).$$

From this and (4.5) in Chapter 1, it follows that

$$\int_{S_k}^{T_k} 1_D(B_t)dt \quad \text{are i.i.d.}$$

Since these random variables have positive mean it follows from the SLLN that

$$\int_0^\infty 1_D(B_t)dt \geq \lim_{n\to\infty} \sum_{k=1}^{n} \int_{S_k}^{T_k} 1_D(B_t)dt = \infty \quad a.s.$$

and hence (2.1) holds.

**Proof of (2.2):** If $f$ is a nonnegative function, then Fubini's theorem implies

$$\begin{aligned} E_x \int_0^\infty f(B_t)dt &= \int_0^\infty E_x f(B_t)dt = \int_0^\infty \int_{\mathbb{R}^d} p_t(x,y)f(y)dydt \\ &= \int_{\mathbb{R}^d} \int_0^\infty p_t(x,y)dtf(y)dy \end{aligned}$$

where $p_t(x,y) = (2\pi t)^{-d/2} e^{-|y-x|^2/2t}$ is the transition density for Brownian motion. As $t \to \infty$, $p_t(x,y) \sim (2\pi t)^{-d/2}$, so if $d \leq 2$

then $\int p_t(x,y)dt = \infty$. When $d \geq 3$, changing variables by letting $t = |y-x|^2/2s$ yields

$$\begin{aligned}
\int_0^\infty p_t(x,y)dt &= \int_\infty^0 \left(\frac{s}{\pi|y-x|^2}\right)^{d/2} e^{-s}\left(-\frac{|y-x|^2}{2s^2}\right) ds \\
&= \frac{|y-x|^{2-d}}{2\pi^{d/2}} \int_0^\infty s^{(d/2)-2} e^{-s} ds \\
&= \frac{\Gamma((d/2)-1)}{2\pi^{d/2}} |y-x|^{2-d}
\end{aligned}$$

where $\Gamma(\alpha) = \int_0^\infty s^{\alpha-1} e^{-s} ds$ is the gamma function. Now define

$$G(x,y) = \int_0^\infty p_t(x,y)dt.$$

Then for $d \geq 3$, $G(x,y) < \infty$ for $x \neq y$, and

$$E_x \int_0^\infty f(B_t)dt = \int G(x,y) f(y) dy.$$

To complete the proof of (2.2) we take $f = 1_D$ with $D = B(0,r)$

and change to polar coordinates. Then

$$\int_D G(0,y)dy = \int_0^r s^{d-1} C_d s^{2-d} ds = \frac{C_d}{2} r^2 < \infty.$$

To extend the last conclusion to $x \neq 0$, observe that applying the strong Markov property at the exit time $\tau$ from $B(0,|x|)$ for a Brownian motion starting at 0 and using the rotational symmetry yields

$$E_x \int_0^\infty 1_D(B_s)ds = E_0 \int_\tau^\infty 1_D(B_s)ds \le E_0 \int_0^\infty 1_D(B_s)ds.$$

We call $G(x,y)$ the potential kernel, because $G(\cdot,y)$ is the electrostatic potential of a unit chage at $y$; see Chapter 3 of Port and Stone (1978) for more. In $d \le 2$, $\int_0^\infty p_t(x,y)dt \equiv \infty$ so we need to take another approach to define a useful $G$:

$$G(x,y) \equiv \int_0^\infty = \int_0^\infty (p_t(x,y) - a_t)dt$$

where the $a_t$ are constants we will choose to make the integral converge (when $x \neq y$). To see why this modified definition might be useful, note that if $\int f(y)dy = 0$, then (assuming that the interchange of integrations can be justified),

$$\int G(x,y)f(y)dy = \int_0^\infty E_x f(B_t)dt.$$

When $d = 1$, we let $a_t = p_t(0,0)$. With this choice

$$G(x,y) = \frac{1}{\sqrt{2\pi}}\int_0^\infty (e^{-(y,x)^2/2t} - 1)t^{-1/2}dt$$

and the integral converges, since the integrand is $\leq 0$ and $\sim (y-x)^2/2t^{3/2}$ as $t \to \infty$ . Changing variables to $t = (y-x)^2/2u$ gives

$$\begin{aligned} G(x,y) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty (e^{-u} - 1)\left(\frac{2u}{(y-x)^2}\right)^{1/2}\frac{-(y-x)^2}{2u^2}du \\ &= \cdots = -\frac{|y-x|}{\sqrt{\pi}}\int_0^\infty s^{-1/2}e^{-s}ds = -|y-x|. \end{aligned}$$

The computation is almost the same for $d = 2$. The only thing that changes is the choice of $a_t$. If we take $a_t = p_t(0,0)$ again, then for $x \neq y$ the integrand behaves asymptotically as $-t^{-1}$ as $t \to 0$, and the integral diverges, so we let $a_t = p_t(0, e_1)$ where $e_1 = (1,0)$. With this choice of $a_t$ we find that

$$\begin{aligned}
G(x,y) &= \frac{1}{2\pi}\int_0^\infty \left(e^{-|x-y|^2/2t} - e^{-1/2t}\right) t^{-1} dt \\
&= \frac{1}{2\pi}\int_0^\infty \left(\int_{|x-y|^2/2t}^{1/2t} e^{-s} ds\right) t^{-1} dt \\
&= \frac{1}{2\pi}\int_0^\infty e^{-s} ds \int_{|x-y|^2/2s}^{1/2s} t^{-1} dt \\
&= \frac{1}{2\pi}\left(\int_0^\infty e^{-s} ds\right)\left(-\log(|x-y|^2)\right) = -\frac{1}{\pi}\log(|x-y|).
\end{aligned}$$

In summary, the potential kernels are given by:

$$G(x,y) = \begin{cases} (\Gamma(d/2-1)/2\pi^{d/2}) \cdot |x-y|^{2-d} & d \geq 3 \\ (-1/\pi) \cdot \log(|x-y|) & d = 2 \\ -1 \cdot |x-y| & d = 1. \end{cases}$$

Note that in each case $G(x,y) = C\varphi(|x-y|)$ where $\varphi$ is the harmonic function we used in Section 3.1. This is, of course, not just coincidence. $x \mapsto G(x,0)$ is clearly spherically symmetric, and (as will be seen), $\Delta G(x,0) = 0$ for $x \neq 0$, so it follows that $G(x,0) = A + B\varphi(|x|)$. The above formulas correspond to $A = 0$. What about the $B$'s? In fact, the $B$'s are chosen so that $(1/2)\Delta G(x,0) = -\delta_0$ (point mass at 0) in the distributional sense. This is easily seen for $d = 1$: in that case $\varphi(x) = |x|$ and then

$$\varphi'(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$$

so if $B = -1$, then $B\varphi''(x) = -2\delta_0$.

# 3: Brownian motion, II: Exit times; Section 3.3

In this section we study the moments of the exit times $\tau = \inf\{t : B_t \notin G\}$ for various open sets $G$. First consider $G = \{x : |x| < r\}$ in which case $\tau = S_r$ in the notation of Section 3.1.

**Theorem 3.1:** If $|x| \le r$, then $E_x S_r = (r^2 - |x|^2)/d$.

**Proof:** The key observation is that since

$$|B_t|^2 - dt = \sum_{j=1}^{d} \{(B_t^i)^2 - t\}$$

is the sum of $d$ martingales it is also a martingale. Using the optional stopping theorem at the bounded stopping time $S_r \wedge t$ we have

$$|x|^2 = E_x\{|B_{S_r \wedge t}|^2 - (S_r \wedge t)d\}.$$

(1.3) tells us that $E_x S_r < \infty$, and we have $|B_{S_r \wedge t}|^2 \le r^2$, so letting $t \to \infty$ and using the dominated convergence theorem gives $|x|^2 = E_x(r^2 - S_r d)$, which implies the desired result. $\square$

To get more formulas like (3.1) we need more martingales. Applying Itô's formula (10.2) in chapter 2, with $X_t^1 = X_t$, a continuous local martingale, and $X_t^2 = \langle X \rangle_t$ we obtain

$$
\begin{aligned}
f(X_t, \langle X \rangle_t) - f(X_0, 0) &= \int_0^t D_1 f(X_s, \langle X \rangle_s) dX_s \\
&\quad + \int_0^t D_2 f(X_s, \langle X \rangle_s) d\langle X \rangle_s \qquad (3.2) \\
&\quad + \frac{1}{2} \int_0^t D_{11} f(X_s, \langle X \rangle_s) d\langle X \rangle_s .
\end{aligned}
$$

From (3.2) we see that if $((1/2)D_{11} + D_2) f = 0$, then $f(X_t, \langle X \rangle)$ is a local martingale. Examples of such functions are

$$f(x,y) = x,\ x^2 - y,\ x^3 - 3xy,\ x^4 - 6x^2 y + 3y^2, \ldots$$

or, to give the general pattern,

$$f_n(x,y) = \sum_{0 \le m \le [n/2]} c_{n,m} x^{n-2m} y^m$$

where $[n/2]$ denotes the largest integer $\le n/2$, $c_{n,0} = 1$ and for $0 \le m < [n/2]$ we take

$$\frac{1}{2}c_{nm}(n-2m)(n-2m-1) = -(m+1)c_{n,m+1}$$

so that $D_{xx}/2$ of the $m$th term is cancelled by $D_y$ of the $(m+1)$th. Note that $D_{xx}$ of the $[n/2]$th term is 0. The first two of these functions give nothing new: ($X_t$ and $X_t^2 - \langle X \rangle_t$ are local martingales), but beyond that we see some new local martingales:

$$X_t^3 - 3X_x\langle X \rangle_t, \quad X_t^4 - 6X_t^2\langle X \rangle_t + 3\langle X \rangle_t^2, \ldots$$

These local martingales are useful for computing expectations for one dimensional Brownian motion.

**Theorem 3.3:** Let $\tau_a \equiv \inf\{t : |B_t| \ge a\}$. Then: (i) $E_0\tau_a = a^2$; (ii) $E_0\tau_a^2 = 5a^4/3$.

The dependence of the moments on $a$ is easy to understand: the Brownian scaling relationship $B_{ct} \stackrel{d}{=} c^{1/2} B_t$ implies that $\tau_a / a^2 \stackrel{d}{=} \tau_1$.

**Proof:** (i) follows from (3.1), so we focus on proving (ii). To do this, let $X_t = B_t^4 - 6B_t^2 t + 3t^2$ and let $T_n \le n$ be stopping times so that $T_n \nearrow \infty$ and $X_{t \wedge T_n}$ is a martingale. Since $T_n \le n$ the optional stopping theorem implies

$$0 = E_0 \{ B_{\tau_a \wedge T_n}^4 - 6 B_{\tau_a \wedge T_n}^2 (\tau - a \wedge T_n) + 3(\tau_a \wedge T_n)^2 \}.$$

Now since $|B_{\tau_a \wedge T_n}| \le a$, so using (1.3) and the dominated convergence theorem we can let $n \to \infty$ to conclude $0 = a^4 - 6a^2 E_0 \tau_a + 3E_0 \tau_a^2$. Using (i) and rearranging yields (ii). $\square$

The next result is a special case of the Burkholder - Davis - Gundy inequalities (5.1), but it is needed to prove (4.4), which is a key step in the proof of (5.1).

**Theorem 3.4:** If $X_t$ is a continuous local martingale with $X_0 = 0$ then

$$E\left(\sup_t X_t^4\right) \le 381 E\langle X\rangle_\infty^2.$$

**Proof:** First suppose that $|X_t|$ and $\langle X\rangle_t$ are $\le M$ for all $t$. In this case (2.5) in Chapter 2 implies $X_t^4 - 6X_t^2\langle X\rangle_t + 3\langle X\rangle_t^2$ is a martingale, so its expectation is 0. Rearranging and using the Cauchy-Schwarz inequality yields

$$EX_t^4 + 3E\langle X\rangle_t^2 = 6E(X_t^2\langle X\rangle_t) \le 6(EX_t^4)^{1/2}(E\langle X\rangle_t^2)^{1/2}.$$

Using the $L^4$ maximal inequality ((4.3) in Chapter 4 of Durrett (1995)) and the fact that $(4/3)^4 \le 3.1605 < 19/6$ it follows that

$$E \sup_{s\le t} X_s^4 \le (4/3)^4 E X_t^4 \le 19 \left( E \sup_{s \le t} X_s^4 \right)^{1/2} (E\langle X\rangle_t^2)^{1/2}.$$

Since $|X_s| \le M$ for all $s$ we can divide each side by $E(\sup_{s\le t} X_s^4)^{1/2}$, and then square to get

$$E \left( \sup_{s\le t} X_s^4 \right) \le 381 (E\langle X\rangle_t^2)^{1/2}.$$

The last inequality holds for a general martingale if we replace $t$ by $T_n = \inf\{t :\ \ t, |X_t|, \text{ or } \langle X\rangle_t \ge n\}$. Using that conclusion, letting $n \to \infty$, and using the monotone convergence theorem yields the claimed inequality. $\square$

Noting that $f(x,y) = \exp(x - y/2)$ satisfies $(\frac{1}{2}D_{11} + D_2)f = 0$ we find the following useful result:

**Theorem (3.5):** If $X$ is a continuous local martingale, then $\mathcal{E}(X)_t = \exp(X_t - \frac{1}{2}\langle X\rangle_t)$ is a local martingale.

If we let $Y_t = \exp(X_t - \frac{1}{2}\langle X\rangle_t)$, then (3.2) says that

$$Y_t - Y_0 = \int_0^t Y_s dX_s \tag{3.6}$$

or, in stochastic differential notation, that $dY_t = Y_t dX_t$. This property gives $Y_t$ the right to be called the "martingale exponential of $Y_t$". As in the case of the ordinary differential equation,

$$f'(t) = f(t)a(t), \qquad f(0) = 1$$

which, for a given continuous function $a(t)$ has the unique solution $f(t) = \exp(A_t)$ where $A_t = \int_0^t a(s)ds$. It is possible to prove (under suitable assumptions) that $Z$ is the only solution of (3.6). See Doleans-Dade (1970) for details.

The exponential local martingale will play an important role in Section 5.3. Then (and now) it will be useful to know when the exponential local martingale is a martingale. The next result is useful even though there are results of this type with weaker conditions; see e.g. Chapter VIII of Revuz and Yor (1991), but it suffices for our present purposes.

**Theorem 3.7:** Suppose $X_t$ is a continuous local martingale with $\langle X\rangle_t \le Mt$ and $X_0 = 0$. Then $\exp(X_t - \frac{1}{2}\langle X\rangle_t)$ is a martingale.

**Proof:** Let $Z_t = \exp(2X_t - \frac{1}{2}\langle 2X\rangle_t)$, which is a local martingale by (3.5). Now

$$Y_t^2 = \exp(2X_t - \langle X\rangle_t) = Z_t \exp(\langle X\rangle_t).$$

Thus if $T_n$ is a sequence of times that reduces $Z_t$, the $L^2-$maximal inequality applied to $Y_{t\wedge T_n}$ gives

$$E\left(\sup_{s\le t} Y^2_{s\wedge T_n}\right) \le 4EY^2_{t\wedge T_n} \le 4e^{Mt}E(Z_{t\wedge T_n}) \le 4e^{Mt}.$$

Letting $n \nearrow \infty$ and using the monotone convergence theorem we have

$$4e^{Mt} \geq E\left(\sup_{s\leq t} Y_s^2\right) \geq \left(E \sup_{s\leq t} |Y_s|\right)^2$$

by Jensen's inequality. Using (2.5) in Chapter 2 we see that $Y_t$ is a martingale. $\square$.

**Remark:** It follows from the last proof that if $X_t$ is a continuous local martingale with $X_0 = 0$ and $\langle X\rangle_t \leq M$ for all $t$, then $Y_t = \exp(X_t - \frac{1}{2}\langle X\rangle_t)$ is a martingale in $\mathcal{M}^2$. (Can $\langle X\rangle_t \leq M$ be replaced by $\langle X\rangle_t \leq Mt$ in this remark?)

Letting $\theta \in \mathbb{R}$ and setting $X_t = \theta B_t$ in (3.6) where $B_t$ is one dimensional Brownian motion, gives us a family of martingales $\exp(\theta B_t - \theta^2 t/2)$. These martingales are useful for computing the distribution of hitting times associated with Brownian motion.

**Theorem 3.8:** Let $T_a = \inf\{t : \ B_t = a\}$. Then for $a > 0$ and $\lambda \geq 0$

$$E_0 \exp(-\lambda T_a) = e^{-a\sqrt{2\lambda}}.$$

**Remark:** If you are good at inverting Laplace transforms, you can use (3.8) to prove (4.1) in Chapter 1:

$$P_0(T_a \leq t) = \int_0^t (2\pi s^3)^{-1/2} a e^{-a^2/2s} ds.$$

**Proof**: $P_0(T_a < \infty) = 1$ by (1.5). Let $X_t = \exp(\theta B_t - \theta^2 t/2)$ and let $S_n \leq n$ be stopping times so that $S_n \nearrow \infty$ and $X_{t \wedge S_n}$ is a martingale. Since $S_n \leq n$, the optional sampling theorem yields

$$1 = E_0 \exp(\theta B_{T_a \wedge S_n} - \theta^2 (T_a \wedge S_n)/2).$$

If $\theta \geq 0$ the right hand side is $\leq \exp(\theta a)$, so letting $n \to \infty$ and using the bounded convergence theorem we have $1 = E_0 \exp(\theta a - \theta^2 T_a/2)$. Taking $\theta = \sqrt{2\lambda}$ now gives the claimed result. $\square$

**Theorem 3.9:** Let $\tau_a \equiv \inf\{t : \ |B_t| \geq a\}$. Then for $a > 0$ and $\lambda \geq 0$,

$$E_0 \exp(-\lambda\tau_a) = 2e^{-a\sqrt{2\lambda}}/(1 + e^{-2a\sqrt{2\lambda}}).$$

**Proof:** Let $\psi_a(\lambda) = E_0(-\lambda T_a)$. Applying the strong Markov property at time $\tau_a$ (and dropping the subscript $a$ to simplify typesetting) gives

$$\begin{aligned} E_0 \exp(-\lambda T_a) &= E_0(\exp(-\lambda\tau) 1_{[B_\tau = a]}) \\ &\qquad + E_0(\exp(-\lambda\tau)\psi_{2a}(\lambda) 1_{[B_\tau = -a]}). \end{aligned}$$

Symmetry considerations imply that $(\tau, B_\tau) \stackrel{d}{=} (\tau, -B_\tau)$. Since $B_\tau \in \{-a, a\}$, it follows that $\tau$ and $B_\tau$ are independent and we have

$$\psi_a(\lambda) = \frac{1}{2}(1 + \psi_{2a}(\lambda)) E_0 \exp(-\lambda\tau).$$

Using the expression for $\psi_a(\lambda)$ given in (3.8) now gives the desired result. $\square$

Here is a bit of calculus that will be useful in other contexts.

**Theorem 3.10:** If $\theta > 0$ then

$$\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2t} e^{-\theta t} dt = \frac{1}{\sqrt{2\theta}} \text{exp}(-|z|\sqrt{2}\theta).$$

**Proof:** Changing variables $t = 1/2s$, $dt = -ds/2s^2$, the integral above

$$\begin{aligned} &= \int_0^\infty \frac{\sqrt{2s}}{\sqrt{2\pi}} e^{-z^2 s} e^{-\theta/2s} \frac{ds}{2s^2} \\ &= \frac{1}{\sqrt{2\theta}} \int_0^\infty \frac{1}{\sqrt{2\pi s^3}} \sqrt{\theta} e^{-\theta/2s} e^{-z^2 s} ds. \end{aligned}$$

Using (3.8) now with $a = \sqrt{\theta}$ and $\lambda = z^2$ and noting the remark after (3.8) for the density function of $T_a$ we see that the last expression is equal to the right side of (3.10). $\square$

The exponential martingale can also be used to study a one-dimensional Brownian motion $B_t$ plus drift. Let $Z_t = \sigma B_t + \mu t$

where $\sigma > 0$ and $\mu \in \mathbb{R}$. Then $X_t = Z_t - \mu t$ is a martingale with $\langle X \rangle_t = \sigma^2 t$ so (3.7) implies that

$$\exp(\theta(Z_t - \mu t)) - \theta^2\sigma^2 t/2)$$

is a martingale. If $\theta = -2\mu/\sigma^2$, then $-\theta\mu - \theta^2\sigma^2/2 = 0$ and $\exp(-(2\mu/\sigma^2)Z_t)$ is a local martingale. Repeating the proof of (3.8) one gets:

**Exercise 3.3:** Let $T_{-a} = \inf\{t : \; Z_t = -a\}$. If $a, \mu > 0$, then

$$P_0(T_{-a} < \infty) = \exp(-2a\mu/\sigma^2).$$

Use this to show that $P_0(\inf_t Z_t < -a) = \exp(-2a\mu/\sigma^2)$. (Compare with GRS PfS Example 7.2, (17), page 374.)