Math/Stat 523, Spring 2020

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Lecture 13

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Outline

- 1: Meyer-Tanaka Formula; Section 2.11
- 2: Girsanov's formula; Section 2.12

1. Meyer-Tanaka Formula; Section 2.11

Our goal in this section is to prove an extension of Itô's formula due to Meyer (1976) who started from work of Tanaka (1963).

Theorem 11.1: Let $f : \mathbb{R} \to \mathbb{R}$ be convex and let X be a continuous semimartingale. Then $f(X)$ is a semimartingale and

$$f(X_t) - f(X_0) = \int_0^t f'(X_s) dX_s + K_t.$$

Here $f'(x) = \lim_{h \downarrow 0}(f(x) - f(x-h))/h$ is the left derivative which exists at all x by convexity and K_t is a continuous adapted increasing process.

Proof: Let $X_t = M_t + A_t$ be a (unique) decomposition of the semimartingale. By stopping we can suppose without loss of generality that $|X_t| \le N$, $|A_t| \le N$, and $\langle M \rangle_t \le N$ for all t. Let g be a C^∞ function with compact support in $[-\infty, 0]$ and having $\int g(s)ds = 1$. Let

$$f_n(x) = \int f(x + \frac{y}{n}) g(y) dy.$$

Then f_n is convex and C^∞. By Itô's formula we have

$$f_n(X_t) - f_n(X_0) = \int_0^t f_n'(X_s)dX_s + \frac{1}{2}\int_0^t f_n''(X_s)dX_s.$$

Since a convex function is Lipschitz continuous on each bounded interval, it is easy to see that $f_n(x) \to f(x)$ uniformly on compact sets. Since $|X_t| \le N$, it follows that

$$f_n(X_t) \to f(X_t) \quad \text{uniformly in} \quad t. \qquad (c)$$

To deal with the first term on the right, we differentiate (a) to find that

$$f_n'(x) = \int f'\left(x + \frac{y}{n}\right) g(y)dy \nearrow f'(x) \qquad (d)$$

as $n \to \infty$. Now if $I_t^n = \int_0^t f_n'(X_s)dM_s$ and $I_t = \int_0^t f'(X_s)dM_s$, then the L^2 maximal inequality for martingales and the isometry property of stochastic integrals (Exercise 6.2) imply

$$\begin{aligned} E\left(\sup_t (I_t^n - I_t)^2\right) &\le 4 \sup_t E(I_t^n - I_t)^2 \\ &= 4E\int_0^\infty (f_n'(X_s) - f'(X_s))^2 d\langle M\rangle_s \to 0 \end{aligned}$$

by the bounded convergence theorem (upon recalling $\langle M\rangle_\infty \le N$). By passing to a subsequence we can improve the last conclusion to

$$\sup_t (I_t^{n_k} - I_t)^2 \to_{a.s.} 0 \quad \text{almost surely.} \qquad (e)$$

Now let $J_t^n = \int_0^t f_n'(X_s) dA_s$ and $J_t = \int_0^t f'(X_s) dA_s$. In this case it follows from the bounded convergence theorem that for almost every ω

$$\sup_t |J_t^n - J_t| \le \int_0^\infty |f_n'(X_s) - f'(X_s)| dA_s \to 0. \qquad (f)$$

To take the limit of the third and final term, $K_t^n = \frac{1}{2}\int_0^t f_n''(X_s)d\langle X\rangle_s$, we note that (b) implies

$$K_t^{n_k} = f_{n_k}(X_t) - f_{n_k}(X_0) - \int_0^t f_{n_k}'(X_s)dX_s. \qquad (g)$$

Combining (c), (e), and (f) we see that almost surely the right hand side of (g) converges to a limit uniformly in t, so the left side converges to a limit uniformly in t as well. Since $K_t^{n_k}$ is continuous, adapted, and increasing, the limit also has these properties. □

If we fix X then for a given f we call K the increasing process associated with f. The increasing process associated with $|x-a|$ is called the local time at a, and is denoted by L_t^a To prove the first property that justifies this name, (11.3) below, we need the following lemma:

Lemma 11.2: The increasing process associated with $(x-a)^+$ or $(x-a)^-$ is $(1/2)L_t^a$.

Proof: Let $f_1(x) = (x-a)^+$, $f_2(x) = (x-a)^-$, and let K_t^i be the increasing process associated with f_i. We begin by observing $f_1 + f_2 = |x-a|$, so $K_t^1 + K_2^2 = L_t^a$. Our second claim is that since $f_1 - f_2 = x - a$, we have $K_t^1 - K_t^2 = 0$. To see this let $g(x) = x - a$ which has $g'(x) = 1$ and note $X_t - X_0 = \int_0^t 1 dX_s$, so the associated K_t must be zero. □

Theorem 11.3: L_t^a increases only when $X_t = a$; or to be precise, if we let ℓ^a be the measure with distribution function $t \mapsto L_t^a$, then ℓ^a is supported by $\{t : \ X_t = a\}$.

Proof: Intuitively, $|X_t - a|$ is a local martingale except when $X_t = a$, so L_t^a is constant when $X_t \neq a$. To prove (11.3), though, it is easer to use the alternative definitions in (11.2). Let $S < T$ be stopping times so that $[S, T] \subset \{t : X_t < a\}$. Applying (11.1) to $f(x) = (x-a)^+$ we have

$$(X_T - a)^+ - (X_S - a)^+ = \int_S^T 1_{[X_s > a]} dX_s + \frac{1}{2}(L_T^a - L_S^a).$$

The left side and the integral on the right side vanish, so $L^a_T = L^a_S$. Since this holds when $S \equiv q$ with a any rational and $T = \inf\{t > S \ : \ X_t \geq a - 1/n\}$ for any n, it follows that $\ell^a(\{t : X_t < a\}) = 0$. A similar argument using $(x-a)^-$ instead of $(x-a)^+$ shows $\ell^a(\{t : X_t > 0\}) = 0$. $\square$

Here is our main result:

Theorem (11.4): Let X be a continuous semimartingale. Let f be the difference of two convex functions, let f' be the left-derivative of f and suppose $f'' = \mu$ in the sense of distribution (i.e. μ has distribution function f'). Then

$$f(X_t) - f(X_0) \ = \ \int_0^t f'(X_s) dX_s + \frac{1}{2} \int L^a_t d\mu(a).$$

Proof: Since f is the difference of two convex functions and the formula is linear in f, we can suppose without loss of generality that f is convex. By stopping we can suppose $|X_t| \leq N$ and $\langle X \rangle_t \leq N$ for all t. Having done this we can let

$$g(x) = \frac{1}{2} \int_{-N}^{N} |x - a| d\mu(a).$$

Since $(f - g)'' = 0$ on $[-N, N]$ it follows that $f(x) - g(x) = a + bx$ for $|x| \leq N$. Since the result is trivial for linear functions, it suffices now to prove the result for g, which is almost trivial. Differentiating the definition of g shows that

$$g'(x) = \frac{1}{2} \int_{-N}^{N} \text{sign}(x - a) d\mu(a).$$

Starting with the definition of the local time

$$|X_t - a| - |X_0 - a| = \int_0^t \text{sign}(X_s - a) dX_s + L_t^a,$$

and then integrating with respect to μ over $[-N, N]$ yields

$$g(X_t) - g(X_0) \;=\; \frac{1}{2}\int_N^N \left(\int_0^t \mathsf{sign}(X_s - a)dX_s\right) d\mu(a) + \frac{1}{2}\int_{-N}^N L_t^a d\mu(a).$$

(11.5) and (11.6) below will justify interchanging the two integrals in the first term on the right side to give

$$g(X_t) - g(X_0) = \int_0^t g'(X_s)dX_s + \frac{1}{2}\int_{-N}^N L_t^a d\mu(a).$$

Since $L_t^a = 0$ for $|a| > N$ (note that $|X_t| \leq N$ and use (11.3)) this proves the result for g and completes the proof of (11.4). $\square$

For the next two results, let S be a set, let $\mathcal{S}$ be a $\sigma-$field of subsets of S, and let μ be a finite measure on $(S, \mathcal{S})$. For our purposes it would be enough to take $S = \mathbb{R}$, but it is just as easy to treat a general set. The first step in justifying the interchange of the order of integration is to deal with a measurability issue: if we have a family $H_t^1(\omega)$ of integrands for $a \in S$, then we can define the integrals $\int_0^t H_s^a dX_s$ to be a measurable function of (a, t, ω)

Lemma (11.5): Let X be a continuous semimartingale with $X_0 = 0$ and let $H_t^a(\omega)$ be bounded and $\mathcal{S} \times \Pi$ measurable. Then there is a $Z(a, t, \omega) \in \mathcal{S} \times \Pi$ such that for $\mu-$almost every a, $Z(a, t, \omega)$ is a continuous version of $\int_0^t H_s^a dX_s$.

The second lemma needed to justify the interchange is the following Fubini type theorem:

Fubini's Theorem 11.6: Let X be a continuous semimartingale, let $H_t^a \in b(\mathcal{S} \times \Pi)$, and let $Z_t^a \in \mathcal{S} \times \Pi$ be a continuous version of $\int_0^t H_s^a dX_s$ for μ–almost every a. Then $Y_t = \int_S Z_t^a d\mu(a)$ is a version of $H \cdot X$ where $H_t = \int_S H_t^a d\mu(a)$. Less formally

$$\int_S \int_0^t H_s^a dX_s d\mu(a) = \int_0^t \int_S H_s^a d\mu(a) dX_s.$$

The following result completes the identification of L_t^a as the local time L_t^a.

Theorem 11.7: Let X be a continuous semimartingale with local time L_t^a. If g is a bounded Borel measurable function then

$$\int_{-\infty}^{\infty} L_t^a g(a) da = \int_0^t g(X_s) d\langle X \rangle_s$$

Proof: Suppose first that g is continuous and let $f \in C^2$ with $f'' = g$. In this case comparing (11.4) with Itô's formula proves the claimed identity. Since the identity holds for any continuous function, using the Monotone Class Theorem proves the result for a bounded measurable g. $\square$

When X is Brownian motion $\mathbb{B}$, the identity in (11.7) becomes

$$\int_{-\infty}^{\infty} L_t^a g(a) da = \int_0^t g(X_s) ds,$$

and with $g = 1_A$ for a Borel set A the identity becomes

$$\int_A L_t^a da = \int_0^t 1_A(X_s) ds \equiv (\text{the total time } \; X_s \in A \text{ up to time } \; t)\,.$$

One of the notable theorems concerning local time is that (there exist versions of) it which are jointly continuous in a and t jointly. For Brownian motion this is due to Trotter (1958).

Theorem 11.8: Let X be a continuous local martingale. There is a version of the process $\{L_t^a : a \in \mathbb{R}, t \geq 0\}$ for which $(a, t) \mapsto L_t^a$ is continuous.

Example 11.1: Joint continuity of the local time process fails for some semimartingales. In particular it does not hold for $X_t = |\mathbb{B}_t|$, which is a semimartingale by (11.2): it is clear that $L_X^a(t) = L_{\mathbb{B}}^a(t) + L_{\mathbb{B}}^{-a}(t)$ for $a > 0$ and $L_X^a(t) = 0$ for $a < 0$. Since $L_B^0(t) \neq 0$ it follows that $a \mapsto L_X^a(t)$ is discontinuous at $a = 0$. A related example is provided by **skew Brownian motion**. This is the process X_t obtained from Brownian motion by giving signs to the excursions of Brownian motion where the signs $\epsilon_i \in \{\pm 1\}$ are independent of $\mathbb{B}$ and $P(\epsilon_i = 1) = p$, $P(\epsilon_i = -1) = (1 - p)$ where $p > 1/2$. The resulting process Y_t has $L_Y^a(t) \to 2pL_{\mathbb{B}}^0(t)$ as $a \searrow 0$ and $L_Y^a(t) \to 2(1-p)L_{\mathbb{B}}^0(t)$ as $a \nearrow 0$. See Walsh (1978) and Harrison and Shepp (1981).

The latter authors show that X_t satisfies the stochastic differential equation given by $X_t = \mathbb{B}(t) + bL_X^0(t)$ where $|b| \le 1$ and $p = (1+b)/2$. Skew Brownian motion is related to oscillating Brownian motion, which also has discontinuous local time; see Keilson and Wellner (1978).

Proof (of 11.8): As usual, by stopping we can suppose that $|X_t| \le N$ and $\langle X \rangle_t \le N$ for all $t \ge 0$. By (11.2)

$$\frac{1}{2}(X_t - a)^+ - (X_0 - a)^+ - \int_0^t 1_{[X_s > a]} dX_s.$$

It is clear that $(a,t) \mapsto (X_t - a)^+ - (X_0 - a)^+$ is continuous, so we only need to study the joint continuity of the stochastic integral

$$I_t^a = \int_0^t 1_{[X_s > a]} dX_s.$$

Fix a time $T < \infty$ and regard $a \mapsto I^a$ as a mapping from $\mathbb{R}$ to $C([0,T],\mathbb{R})$, the real-valued functions continuous on $[0,T]$, which we equip with the sup norm $\|f\| = \sup_{0 \le t \le T} |f(s)|$. In view of Kolmogorov's continuity theorem ((1.6) in Chapter 1), it suffices to show that for some $\alpha, \beta > 0$ we have

$$E\|I^a_\cdot - I^b_\cdot\|^4 \le C|a-b|^{1+\alpha}.$$

Suppose without loss of generality that $a < b$. One of the Burkholder - Davis - Gundy inequalities ((5.1) in Chapter 3) implies that

$$E\|I^a_\cdot - I^b_\cdot\|^4 \le CE\left(\int_0^T 1_{[a<X_s\le b]} d\langle X_s\rangle_s\right)^2$$

To bound the right side we note that by using (11.7) and then the Cauchy-Schwarz inequality and Fubini's theorem:

$$
\begin{aligned}
&= E\left(\int_a^b L_T^x dx\right)^2 \le (b-a)E\int_a^b (L_T^x)^2 dx \\
&= (b-a)\int_a^b E((L_T^x)^2 dx \le (b-a)^2 \sup_{a\le x\le b} E(L_T^x)^2.
\end{aligned}
$$

To bound the supremum, we recall the definition of L_T^x:

$$L_T^x = |X_T - x| - |X_0 - x| - \int_0^T \text{sign}(X_s - x)dX_s$$

and note that since $(a+b)^2 \le 2(a^2+b^2)$ and $|y-x|-|z-x| \le |y-z|$, we find that

$$
\begin{aligned}
E(L_T^x)^2 &\le 2E(X_T - X_0)^2 + 2\left(\int_0^T \text{sign}(X_s - x)dX_s\right)^2 \\
&\le 2(2N)^2 + 2E(\langle X\rangle_T) \le 8(N^2+N)
\end{aligned}
$$

by the isometry property and the bounds imposed on $|X_t|$ and

$\langle X \rangle_t$ by stopping. Thus the inequality needed for Kolmogorov's continuity theorem holds and the proof is complete. $\square$

2. Girsanov's Formula; Section 2.12

Here we will show that semimartingales and stochastic integrals are not affected by a locally equivalent change of measure. For concreteness the results are phrased in terms of the canonical probability space $(C, \mathcal{C})$ with $\mathcal{F}_t$ the filtration generated by the coordinate maps $X_t(\omega) = \omega_t$. Two measures Q and P defined on a filtration $\mathcal{F}_t$ are said to be **locally equivalent** if for each t their restrictions to $\mathcal{F}_t$, Q_t and P_t, are equivalent; i.e. mutually absolutely continuous. In this case we let $\alpha_t = dQ_t/dP_t$. The reasons for our interest in this quantity will become more clear as the story develops.

Example 12.0: Let $X_t = \mathbb{B}_t$ be standard Brownian motion under P and let $X_t = \mathbb{B}_t + \int_0^t f(s)ds$ for some function $f \in L_2(\mathbb{R}^+)$ under Q. Then α given by

$$\alpha_t \equiv \frac{dQ_t}{dP_t} = \exp\left(\int_0^t f(s)d\mathbb{B}(s) - \frac{1}{2}\int_0^t f^2(s)ds\right). \qquad (12.0)_a$$

is a martingale.

Lemma 12.1: Y_t is a (local) martingale/Q if and only if $\alpha_t Y_t$ is a (local) martingale/P.

Proof: The parentheses are meant to indicate that the statement is true if the two "locals" are removed. Let Y_t be a martingale / Q, let $s < t$, and let $A \in \mathcal{F}_s$. Now if $Z \in \mathcal{F}_t$, then: (i) $\int Z dP = \int Z dP_t$, and (ii) $\int Z \alpha_t dP_t = \int Z dQ_t$. So using (i), (ii), the fact that Y_t is a martingale/Q, (ii) and (i) we have

$$\begin{aligned}\int_A \alpha_t Y_t dP &= \int_A \alpha_t Y_t dP_t = \int_A Y_t dQ_t \\ &= \int_A Y_s dQ_s = \int_A \alpha_s Y_s dP_s = \int_A \alpha_s Y_s dP.\end{aligned}$$

This shows that $\alpha_t Y_t$ is a martingale/P. If Y is a local martingale/Q, then there is a sequence of stopping time $T_n \nearrow \infty$ so that $Y_{t\wedge T_n}$ is a martingale/Q and hence $\alpha_{t\wedge T_n}$ is a martingale/P. The optional sampling theorem implies that $\alpha_{t\wedge T_n} Y_{t\wedge T_n}$ is a martingale/P, and it follows that $\alpha_t Y_t$ is a local martingale/P.

To prove the converse, observe that: (a) interchanging the roles of P and Q and applying the last result shows that if $\beta_t = dP_t/dQ_t$, and Z_t is a (local) martingale/P, then $\beta_t Z_t$ is a (local) martingale/Q and
(b) $\beta_t = \alpha_t^{-1}$, so letting $Z_t = \alpha_t Y_t$ yields the claimed result. $\square$

Since 1 is a martingale/Q we have:

Corollary 12.2: $\alpha_t = dQ_t/dP_t$ is a martingale/P.

Here is a converse of (12.2) which is useful in constructing examples.

Lemma 12.3: Given α_t, a nonnegative martingale/P, there is a unique locally equivalent probability measure Q so that $dQ_t/dP_t = \alpha_t$.

Proof: The last equation in the lemma defines the restriction of Q to $\mathcal{F}_t$ for any t. To see that this defines a unique measure on $\mathcal{C}$, let $t_1 < t_2 < \cdots < t_n$, A_i be Borel subsets of $\mathbb{R}^d$, and let $B = \{\omega : \omega(t_i) \in A_i \text{ for } 1 \leq i \leq n\}$. Define the finite dimensional distributions of a measure Q by setting

$$Q(B) = \int_B \alpha_t dP$$

whenever $t \geq t_n$. The martingale property of α_t implies that the finite-dimensional distributions are consistent so we have defined a unique measure on $(C, \mathcal{C})$. $\square$

Here is the main result of this section:

Girsanov's formula (12.4): If X_t is a local martingale/P, and we let $A_t = \int_0^t \alpha_s^{-1} d\langle \alpha, X\rangle_s$, then $X_t - A_t$ is a local martingale/Q.

Proof: Although the formula for A is a bit strange at first sight, it is easy to see that it must be the right answer. If we suppose that A is locally of b.v. and has $A_0 = 0$, then integrating by parts, i.e. using (10.1) and noting that $\langle \alpha, A\rangle_t = 0$ since A has bounded variation, yields

$$\begin{aligned} &\alpha_t(X_t - A_t) - \alpha_0 X_0 \qquad (12.5) \\ &= \int_0^t (X_s - A_s) d\alpha_s + \int_0^t \alpha_s dX_s - \int_0^t \alpha_s dA_s + \langle \alpha, X\rangle_t \end{aligned}$$

Here we need the assumption made in Section 2.2 that our filtration only admits continuous martingales, so there is a continuous version of α and hence our integration by parts formula applies.

Since α_s and X_s are local martingales/P, the first two terms on the right side in (12.5) are local martingales/P. In view of

(12.1), if we want $X_t - A_t$ to be a local martingale/Q, we need to choose A so that the sum of the third and fourth terms is identically 0; that is

$$\int_0^t \alpha_s dA_s = \langle \alpha, X \rangle_t.$$

But this equation together with the associative law (9.6), it is clear that it is necessary and sufficient that

$$A_t = \int_0^t \alpha_s^{-1} d\langle \alpha, X \rangle_s.$$

The last detail remaining is to prove that the integral defining A in the last display exists. Let $T_n \equiv \inf\{t : \alpha_t \le n^{-1}\}$. If $t \le T_n$, then

$$\int_0^t \frac{d|\langle \alpha, X \rangle|_s}{\alpha_s} \le n \int_0^t d|\langle \alpha, X \rangle|_s < \infty$$

by the Kunita - Watanabe inequality. So, if $T \equiv \lim_{n\to\infty} T_n$, then A_t is well-defined for $t \le T$. By optional sampling $E\alpha_{t \wedge T_n} = E\alpha_t$, so noting that $\alpha_{t\wedge T_n} = \alpha_t$ on $T_n > t$ we have

$$E(\alpha_t 1_{[T_n \le t]}) = E(\alpha_{T_n} 1_{[T_n \le t]}).$$

Since $\alpha_t \ge 0$ is continuous and $T_n \le t$, it follows that

$$E(\alpha_t 1_{[T \le t]}) \le E(\alpha_t 1_{[T_n \le t]} = E(\alpha_{T_n} 1_{[T_n \le t]}) \le 1/n.$$

Letting $n \to \infty$ we see that $\alpha_t = 0$ a.s. on $[T \le t]$, and hence

$$Q_t(T \le t) = E(\alpha_t 1_{[T \le t]}) = 0.$$

But P_t is equivalent to Q_t, so $0 = P_t(T \le t) = P(T \le t) = 0$. Since t is arbitrary, $P(T < \infty) = 0$, and the proof is complete. $\square$

Girsanov's theorem (12.4) shows that the collection of semimartingales is not affected by change of measure. Our next goal is to show that if X is a semimartingale/P, Q is locally equivalent to P, and $H \in \ell b\Pi$, then the integral $(H \cdot X)_t$ is the same under P and Q. Our first step in this direction is given by the following theorem:

Theorem 12.6: The quadratic variation $\langle X \rangle_t$ and hence the covariation $\langle X, Y \rangle_t$ is the same under P and Q.

Proof: The second conclusion follows from the first and the definition of the covariation process (via polarization) given in (3.9).

To prove the first, we recall that (8.6) shows that if $\Delta_n = \{0 = t_0^n < t_1^n < \cdots < t_{k_n}^n = t\}$ is a sequence of partitions of $[0, t]$ with mesh $|\Delta_n| \to 0$ then

$$\sum_i (X_{t_{i+1}^n} - X_{t_i^n})^2 \to \langle X \rangle_t$$

in probability for any semi-martingale/P. Since convergence in probability is not affected by locally equivalent change of measure, the desired conclusion follows. $\square$

Theorem 12.7: If $H \in \ell b\Pi$, then $(H \cdot X)_t$ is the same under P and Q.

Proof: The value is clearly the same for simple integrands. Let M and N be the local martingale parts, and let A and B be the locally bounded variation parts of X under P and Q respectively. Note that (12.6) implies $\langle M \rangle_t = \langle N \rangle_t$. Let

$$T_n = \inf\{t : \ \langle M \rangle_t, \ |A|_t, \text{ or } |B|_t \geq n\}.$$

If $H \in \ell b\Pi$ and $H_t = 0$ for $t \geq T_n$, then by (4.5) we can find simple H^m so that

$$\|H^m - H\|_M,\ \|H^m - H\|_N \to 0 \quad \text{and} \quad \int |H^m_s - H_s| d(|A| + |B|) \to 0.$$

These conditions allow us to pass to the limit in the equality

$$(H^m \cdot (M + A))_t = (H^m \cdot (N + B))_t$$

where the left side is computed under P and the right is computed under Q to conclude that for any $H \in \ell b\Pi$ the integrals under P and Q agree up to time T_n. Since n is arbitrary and $T_n \to \infty$, the proof is complete. □

Example 12.0, continued: Since α_t is a martingale, it follows from Lemma (11.3) that there is a measure Q for which $dQ_t/dP_t = \alpha_t$. Now Girsanov's theorem (12.4) implies that under Q the process $X_t - A_t$ is a local martingale where A_t is given by $A_t = \int_0^t \alpha_s^{-1} d\langle \alpha, X\rangle_t$. But I claim that

$$\begin{aligned}
\langle \alpha, X\rangle_t &= \langle \alpha, \mathbb{B}\rangle_t = \left\langle \int_0^{\cdot} \alpha_s f(s) dX_s, X \right\rangle_t \\
&= \left\langle \int_0^{\cdot} \alpha_s f(s) d\mathbb{B}_s, \mathbb{B} \right\rangle_t \\
&= \int_0^t \alpha_s f(s) d\langle \mathbb{B}\rangle_s = \int_0^t \alpha_s f(s) ds.
\end{aligned}$$

It follows that

$$A_t = \int_0^t \alpha_s^{-1} \alpha_s f(s) ds = \int_0^t f(s) ds.$$

and hence $X_t - A_t$ is a martingale/Q. But $\langle X\rangle_t = \langle \mathbb{B}\rangle_t = t$, and hence $X_t - \int_0^t f(s) ds$ is a Brownian motion process/Q.

Example 12.0, continued; linear drift: Suppose that $\mathbb{B}_\mu$ is Brownian motion with drift $\mu > 0$; this is the special case of Example 12.0 with $f(t) = \mu$. $\tau \equiv \inf\{t > 0 : \mathbb{B}_\mu(t) = a\}$, $a > 0$. Use the result of problem 2 together with results from Can we use Girsanov's theorem class concerning the distribution of τ when $\mu = 0$ to find the distribution of τ when $\mu > 0$? I claim that

$$\begin{aligned} P_\mu(\tau > t) &= P_\mu(\mathbb{B}_\mu(s) < a,\ 0 \le s \le t) \\ &= \Phi\left(\frac{a - \mu t}{\sqrt{t}}\right) - e^{2\mu}\Phi\left(\frac{-a - \mu t}{\sqrt{t}}\right) \end{aligned}$$

and

$$f_\tau(t) = \frac{a}{\sqrt{2\pi t^3}}\exp\left(-\frac{(a-\mu t)^2}{2t}\right) \qquad \text{for } t \ge 0\,.$$

This is the *inverse Gaussian* density. [Note that this reduces to the density of τ_a obtained earlier from a reflection argument when $\mu = 0$.]

First proof: Here is a solution using the martingale property of α which involves almost no calculation:

$$\begin{aligned}
P_\mu(\tau_\mu \le t) &= \int_{[\tau_\mu \le t]} dP_\mu = \int_{[\tau \le t]} \alpha(t) dP_0 \\
&= \int_{[\tau \le t]} \alpha(\tau) dP_0 \\
&\qquad \text{by optional sampling since } \tau \text{ is } \mathcal{A}_\tau - \text{measurable} \\
&= \int_{[\tau \le t]} \text{exp}(\mu a - (1/2)\mu^2 \tau) dP_0 \qquad \text{since } \mathbb{B}_\tau) = a \\
&= \int_{[s \le t]} \text{exp}(\mu a - (1/2)\mu^2 s) \frac{a}{\sqrt{2\pi s^3}} \text{exp}\left(-\frac{a^2}{2s}\right) ds \\
&\qquad \text{by the computation of the density of } \tau \text{ under } P_0 \\
&\qquad \text{as derived before} \\
&= \int_0^t \frac{a}{\sqrt{2\pi s^3}} \text{exp}\left(-\frac{(a-\mu s)^2}{2s}\right) ds \,.
\end{aligned}$$

Thus under P_μ the stopping time $\tau = \tau_a$ has the *inverse Gaussian* density

$$f_\tau(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{(a-\mu t)^2}{2t}\right), \quad t > 0 .$$

Second proof: This argument makes use of the joint density of $\mathbb{B}(t)$ and $M^+ \equiv \sup_{0 \le s \le t} \mathbb{B}(s)$ obtained by a reflection argument:

$$\begin{aligned} f(x,y) &\equiv -\frac{\partial^2}{\partial x \partial y} P(\sup_{0\le s\le t} P(\mathbb{B}(s) \ge x, \mathbb{B}(t) \le y) \\ &= \sqrt{\frac{2}{\pi t^3}}(2x-y)\exp\left(-\frac{(2x-y)^2}{2t}\right) \quad \text{for} \quad 0 \le y \le x. \end{aligned}$$

By using this joint density in the fifth line,

$$
\begin{aligned}
P_\mu(\tau > t) &= P_\mu(M_\mu(t) \equiv \sup_{0\le s\le t} \mathbb{B}_\mu(s) < a) \\
&= \int_{[M_\mu(t)<a]} dP_\mu \\
&= \int_{[M(t)<a]} Y(t)dP_0 \\
&= \int_{[M(t)<a]} \exp(\mu\mathbb{B}(t) - (1/2)\mu^2 t)dP_0 \\
&= \int_{x=0}^{a} \int_{y=-\infty}^{x} \exp(\mu y - (1/2)\mu^2 t) \\
&\qquad \cdot\sqrt{\frac{2}{\pi t^3}}(2x-y)\exp\left(-\frac{(2x-y)^2}{2t}\right) dydx \\
&= \Phi\left(\frac{a-\mu t}{\sqrt{t}}\right) - e^{2\mu a}\Phi\left(\frac{-a-\mu t}{\sqrt{t}}\right),
\end{aligned}
$$

and a differentiation to get the density completes the proof.

The last equality here is checked as follows: first write

$$
\begin{aligned}
&\int_{x=0}^{a}\int_{y=-\infty}^{x}\exp(\mu y-(1/2)\mu^2 t)\\
&\qquad\qquad\cdot\sqrt{\frac{2}{\pi t^3}}(2x-y)\exp\left(-\frac{(2x-y)^2}{2t}\right)dydx\\
&\equiv \int_{x=0}^{a}\int_{y=-\infty}^{x} g(x,y;t,\mu)dydx\\
&= \int_{x=0}^{a}\int_{y=-\infty}^{0} g(x,y;t,\mu)dydx\\
&\qquad + \int_{x=0}^{a}\int_{y=0}^{x} g(x,y;t,\mu)dydx\\
&= \int_{y=0}^{a}\int_{x=y}^{a} g(x,y;t,\mu)dxdy\\
&\qquad + \int_{y=-\infty}^{0}\int_{x=0}^{a} g(x,y;t,\mu)dxdy\\
&\equiv I+II\,.
\end{aligned}
$$

Now

$$\begin{aligned}
I &= \int_{y=0}^{a}\int_{x=y}^{a} g(x,y;t,\mu)\,dx\,dy \\
&= \int_{y=0}^{a}\left(\int_{x=y}^{a}\frac{(2x-y)}{\sqrt{t}}\text{exp}\left(-\frac{(2x-y)^2}{2t}\right)\frac{1}{\sqrt{t}}dx\right) \\
&\qquad\qquad \cdot\text{exp}(\mu y-(1/2)\mu^2 t)\sqrt{\frac{2}{\pi t}}dy \\
&= \int_{y=0}^{a}\left(\text{exp}\left(-\frac{y^2}{2t}\right)-\text{exp}\left(-\frac{(2a-y)^2}{2t}\right)\right) \\
&\qquad\qquad \cdot\text{exp}(\mu y-(1/2)\mu^2 t)\frac{1}{2}\sqrt{\frac{2}{\pi t}}dy \\
&= P(0\le N(\mu t,t)\le a)-e^{2\mu a}P(0\le N(2a+\mu t,t)\le a)\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
II &= \int_{y=-\infty}^{0}\int_{x=0}^{a} g(x,y;t,\mu)dxdy \\
&= \int_{y=-\infty}^{0}\left(\int_{x=0}^{a}\frac{(2x-y)}{\sqrt{t}}\text{exp}\left(-\frac{(2x-y)^2}{2t}\right)\frac{1}{\sqrt{t}}dx\right) \\
&\qquad \cdot\text{exp}(\mu y-(1/2)\mu^2 t)\sqrt{\frac{2}{\pi t}}dy \\
&= \int_{y=-\infty}^{0}\left(\text{exp}\left(-\frac{y^2}{2t}\right)-\text{exp}\left(-\frac{(2a-y)^2}{2t}\right)\right) \\
&\qquad \cdot\text{exp}(\mu y-(1/2)\mu^2 t)\frac{1}{2}\sqrt{\frac{2}{\pi t}}dy \\
&= P(-\infty < N(\mu t,t)\le 0) - e^{2\mu a}P(-\infty < N(2a+\mu t,t)\le 0)\,.
\end{aligned}$$

Hence

$$
\begin{aligned}
I + II &= P(-\infty < N(\mu t, t) \leq a) - e^{2\mu a} P(-\infty < N(2a + \mu t, t) \leq a) \\
&= \Phi\left(\frac{a - \mu t}{\sqrt{t}}\right) - e^{2\mu a} \Phi\left(\frac{-a - \mu t}{\sqrt{t}}\right).
\end{aligned}
$$

Remark 1: Example 12.0, Brownian motion with deterministic drift, was treated by Cameron and Martin (1944). See Kallenberg (1997), Theorem 16.22.

Remark 2: See Steele (2001), Section 13.5, for a treatment of Girsanov type theorems involving both drift and "volatility".

Remark 3: See Steele (2001) for an instructive treatment of Brownian motion with linear drift.

Remark 4: Also see Karatzas and Shreve (1988) Theorem 3.5.1, page 191.