Math/Stat 523, Spring 2020

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Lecture 11

Wednesday May 6, Friday May 8

Outline

- 1: Integration w.r.t. bounded martingales
  Section 2.4.
- 2: Kunita-Watanabe inequality; integration w.r.t. local martingales.
  Sections 2.5 and 2.6.
- 4: Ito's formula ; integration w.r.t. semimartingales Sections 2.7 and 2.8

1. Integration w.r.t. bounded martingales

Here we will develop integrals of predictable processes w.r.t. bounded continuous martingales. When we have the Kunita-Watanabe inequality in Section 2.5 we will be able to extend our integrals to general continuous local martingales.

Much as in the theory of Lebesgue integration we will proceed in several steps:
(i) We will begin with the simplest functions and then extend to the general case by taking limits;
(ii) the sequence of steps used in defining the integrals will also be useful in our further proofs.

Step 1: Basic Integrands

We say $H(s,\omega)$ is a **basic predictable process** if $H(s,\omega) = 1_{(a,b]}(s)C(\omega)$ where $C \in \mathcal{F}_a$. Let $\Pi_0 =$ the set of basic predictable processes. If $H = 1_{(a,b]}C$ and X is continuous, then it is clear that we should define

$$\int H_s dX_s = C(\omega)(X_b(\omega) - X_a(\omega)).$$

Here we restrict our attention to integrating over $[0,\infty)$, since once we know how do that, then we can define the integral over $[0,t]$ by

$$(H \cdot X)_t \equiv \int_0^t H_s dX_s = \int H_s 1_{[0,t]}(s) dX_s.$$

To extend our integral we will need to keep proving versions of the next three results. Under suitable assumptions on H, K, X, and Y:

(a) $(H \cdot X)_t$ is a continuous local martingale.

(b) $((H+K) \cdot X)_t = (H \cdot X)_t + (K \cdot X)_t$,
$(H \cdot (X+Y))_t = (H \cdot X)_t + (H \cdot Y)_t$.

(c) $\langle H \cdot X, K \cdot Y \rangle_t = \int_0^t H_s K_s d\langle X, Y \rangle_s$.

In the current first step, we will only prove a version of (a). (b) and (c) will appear in Step 2. Recall that we define H_t to be bounded if there is an M so that with probability one $H_t \le M$ for all $t \ge 0$.

Theorem 4.1.a: If X is a continuous martingale and $H \in b\Pi_0 = \{H \in \Pi_0 : \ H \text{ is bounded}\}$, then $(H \cdot X)_t$ is a continuous martingale.

Proof: First note that

$$(H \cdot X)_t = \begin{cases} 0, & 0 \le t \le a \\ C(X_t - X_a), & a \le t \le b \\ C(X_b - X_a), & b \le t < \infty. \end{cases}$$

Thus it is clear that $(H \cdot X)$ is continuous, $(H \cdot X)_t \in \mathcal{F}_t$, and $E|(H \cdot X)_t| < \infty$. Since $(H \cdot X)_t$ is constant for $t \notin [a, b]$ we can check the martingale property by considering only $a \le s < t \le b$. In this case

$$\begin{aligned} E((H \cdot X)_t | \mathcal{F}_s) - (H \cdot X)_s &= E((H \cdot X)_t - (H \cdot X)_s | \mathcal{F}_s) \\ &= E(C(X_t - X_s) | \mathcal{F}_s) \\ &= CE(X_t - X_s | \mathcal{F}_s) = 0, \end{aligned}$$

where the last two equalities follows from $C \in \mathcal{F}_s$ and $E(X_t | \mathcal{F}_s) = X_s$. □

Step 2: Simple Integrands

We say $H(s,\omega)$ is a **simple predictable process** and write $H \in \Pi_1$ if H can be written as a sum of a finite number of basic predictable processes. It is easy to see that if $H \in \Pi_1$ then it can be written as

$$H(s,\omega) = \sum_{i=1}^{m} 1_{(t_{i-1},t_i]}(s) C_i(\omega)$$

where $t_0 < t_1 < \cdots < t_m$ and $C_i \in \mathcal{F}_{t_{i-1}}$. In this case we let

$$\int H_s dX_s = \sum_{i=1}^{m} C_i (X_{t_i} - X_{t_{i-1}}).$$

The representation in the definition of H is not unique, since one may subdivide the intervals into two or more pieces, but it is easy to see that the right-side does not depend on the representation of H chosen.

Theorem 4.2.b Suppose that X and Y are continuous martingales. If $H, K \in \Pi_1$ then

$$((H+K)\cdot X)_t = (H\cdot X)_t + (K\cdot X)_t,$$
$$(H\cdot (X+Y))_t = (H\cdot X)_t + (H\cdot Y)_t.$$

Proof: Let $H^1 \equiv H$ and $H^2 \equiv K$. By subdividing the intervals in the definitions we can suppose $H_s^j = \sum_{i=1}^m 1_{(t_{i-1},t_i]}(s) C_i^j$ for $j = 1, 2$. In this case it is easy to see that each side of the first identity is equal to

$$\sum_{i=1}^m (C_i^1 + C_i^2)(X_{t_i} - X_{t_{i-1}}).$$

If $H_s = \sum_{i=1}^m 1_{(t_{i-1},t_i]}(s) C_i$, then each side of the second identity is

$$\sum_{i=1}^m C_i \left\{ (X_{t_i} - X_{t_{i-1}}) + (Y_{t_i} - Y_{t_{i-1}}) \right\}. \qquad \square$$

Since the sum of a finite number of continuous martingales is a continuous martingale, it follows from (4.1.a) and (4.2.b) that the following theorem holds:

Theorem (4.2.c) If X and Y are bounded continuous martingales and $H, K \in b\Pi_1$, then

$$\langle H \cdot X, K \cdot Y \rangle_t = \int_0^t H_s K_s d\langle X, Y \rangle_s.$$

Consequently, $E((H \cdot X)_t (K \cdot Y)_t) = E \int_0^t H_s K_s d\langle X, Y \rangle_s$ and

$$E(H \cdot X)_t^2 = E \int_0^t H_s^2 d\langle X \rangle_s.$$

Remark: Here and in the following, integrals with respect to $\langle X, Y \rangle_s$ are Lebesgue-Stieltjes integrals. That is, since $s \mapsto \langle X, Y \rangle_s$ is locally of bounded variation it defines a $\sigma-$finite measure, so we fix an ω and then integrate with respect to that measure. In the case under consideration the measure is piece-wise constant so the integral is trivial.

Proof: To prove these results it suffices to show that

$$Z_t = (H \cdot X)_t (K \cdot Y)_t - \int_0^t H_s K_s d\langle X, Y \rangle_s$$

is a martingale, for then the first result follows from (3.11), the second follows by taking expected values, and the third formula follows from the second by taking $H = K$ and $X = Y$. To prove that Z_t is a martingale, we begin by noting that

$$\begin{aligned} ((H^1 + H^2) \cdot X)_t (K \cdot Y)_t &= (H^1 \cdot X)_t (K \cdot Y)_t + (H^2 \cdot X)_t (K \cdot Y)_t, \\ \int_0^t (H_s^1 + H_s^2) K_s d\langle X, Y \rangle_s &= \int_0^t H_s^1 K_s d\langle X, Y \rangle_s + \int_0^t H_s^2 K_s d\langle X, Y \rangle_s. \end{aligned}$$

The last two observations imply that if the result holds for (H^1, K) and (H^2, K), then it holds for $(H^1 + H^2), K)$. Similarly, if the result holds for (H, K^1) and (H, K^2), then it holds for (H, K^1) and (H, K^2).

In view of the results in the last paragraph, we can now prove the result by establishing it in the case $H = 1_{(a,b]} C$, $K = 1_{(c,d]} D$, and moreover we can assume that (i) $b \le c$, or (ii) $a = c$, $b = d$.

Case i. In this case $\int_0^t H_s K_s d\langle X,Y\rangle_s \equiv 0$, so we need to show that $(H\cdot X)_t(K\cdot Y)_t$ is a martingale. To prove this, note that if $J = C(X_b - X_a)D1_{(c,d]}$, then $(H\cdot X)_t(K\cdot Y)_t = (J\cdot Y)_t$ is a martingale by (4.1.a).

Case ii. In this case

$$Z_s = \begin{cases} 0 & s \le a \\ CD\{(X_s - X_a)(Y_s - Y_a) - (\langle X,Y\rangle_s - \langle X,Y\rangle_a)\} & a \le s \le b \\ CD\{(X_b - X_a)(Y_b - Y_a) - (\langle X,Y\rangle_b - \langle X,Y\rangle_a)\} & s \ge b, \end{cases}$$

so it suffices to check the martingale property for $a \le s \le t \le b$. To do this, note that

$$\begin{aligned} Z_t - Z_s &= CD\{X_tY_t - X_sY_s - X_a(Y_t - Y_s) - Y_a(X_t - X_s) \\ &\quad - (\langle X,Y\rangle_t - \langle X,Y\rangle_s)\}. \end{aligned}$$

Taking the expected value conditional on $\mathcal{F}_s$ and noting that $X_a \in \mathcal{F}_s$, and $E(Y_t - Y_s|\mathcal{F}_s) = 0$, we have

$$E(Z_t - Z_s|\mathcal{F}_s) = CDE(X_tY_t - \langle X, Y\rangle_t - \{X_sY_s - \langle X, Y\rangle_s\}|\mathcal{F}_s) = 0.$$

This completes the proof of case (ii) and finishes the proof of (4.2.c). □

Step 3: Square integrable integrands, bounded martingales

Now let $\Pi_2(X)$ be the set of all predictable processes H that have

$$\|H\|_X \equiv \left(E\int H_s^2 d\langle X\rangle_s\right)^{1/2} < \infty.$$

Exercise 4.2: $\|H\|_X$ is a norm.

Remark: If we define a measure on the predictable $\sigma-$field by

$$\mu(A \times (s,t]) = E\{(\langle X\rangle_t - \langle X\rangle_s)1_A\},$$

when $A \in \mathcal{F}_s$, then $\Pi_2(X) = L^2(\Pi, \mu)$. μ is called the **Doléans** measure after C. Doléans-Dade. See Chung and Williams (1990), pages 52-53 for a proof that this is a measure.

Now let $\mathcal{M}^2$ be the set of all martingales adapted to $\{\mathcal{F}_t,\ t \geq 0\}$ that have

$$\|X\|_2 = (\sup_t EX_t^2)^{1/2} < \infty.$$

In the proof of (4.6) it will be shown that $\mathcal{M}^2$ is isomorphic to $L^2(\mathcal{F}_\infty)$.

Exercise 4.3: $X \in \mathcal{M}^2$ if and only if $EX_0^2 < \infty$ and $E\langle X\rangle_\infty < \infty$.

The following is the key to our next extension:

(4.4) Isometry Property: If X is a bounded martingale and $H \in b\Pi_1$, then $\|H \cdot X\|_2 = \|H\|_X$.

Reminder: the classes Π_0, $b\Pi_0$, Π_1, $b\Pi_1$, and Π_2.

- $\Pi_0 \equiv \{\text{all basic predictable processes}\}$
  $= \{H(s,\omega) = 1_{(a,b]}(s)C(\omega),\ C \in \mathcal{F}_a\}$.
- $b\Pi_0 \equiv \{H \in \Pi_0 :\ H \text{ is bounded}\}$.
- $\Pi_1 \equiv \{H = \sum_1^m H_j :\ H_j \in \Pi_0\}$.
- $b\Pi_1 \equiv \{H \in \Pi_1 :\ H \text{ is bounded}\}$.
- $\Pi_2(X) \equiv \{H \text{ predictable } \|H\|_X < \infty\}$; $\|H\|_X^2 \equiv E \int_0^\infty H_s^2 d\langle X\rangle_s$.

Proof: Recalling the definitions and using the last formula in (4.2.c) gives

$$\begin{aligned} \|H\|_X^2 &= E\int H_s^2 d\langle X\rangle_s = \sup_t E\int_0^t H_s^2 d\langle X\rangle_s \\ &= \sup_t E(H\cdot X)_t^2 = \|H\cdot X\|_2^2. \end{aligned} \qquad \square$$

To extend the integral to Π_2 we need to show that $b\Pi_1$ is dense in $\Pi_2(X)$. The following lemma proves a slightly stronger result.

Lemma (4.5): Suppose that $X^i \in \mathcal{M}^2$ and $H \in \Pi_2(X^i)$ for $1 \le i \le k$. Then there is a sequence $H^n \in b\Pi_1$ with $\|H^n - H\|_{X^i} \to 0$ for $1 \le i \le k$.

Proof: Since $X^i \in \mathcal{M}^2$, $E\langle X^i \rangle_t < \infty$ and it follows that $b\Pi_1 \subset \Pi_2(X^i)$. Let $\mathcal{H}_t$ be the collection of predictable G that vanish on (t, ∞) for which the conclusion holds. Clearly, if $r < s \leq t$ and $A \in \mathcal{F}_r$, then $G = 1_{(r,s]} 1_A \in \mathcal{H}_t$. Now suppose that $0 \leq G_n \in \mathcal{H}_t$ and $G_n \nearrow G$ with G bounded. The dominated convergence theorem gives

$$\|G - G_n\|_{X^i}^2 = E \int (G_s - G_s^n)^2 d\langle X \rangle_s^i \to 0$$

as $n \to \infty$, so we can pick n so that the last difference is $< \epsilon^2$ for $1 \leq i \leq k$. Since $G_n \in \mathcal{H}_t$, we can find a sequence $H^{n,m} \in b\Pi_1$ so that $\|H^{n,m} - G^n\|_{X^i} \to 0$ for $1 \leq i \leq k$ and then pick m so that $\|H^{n,m} - G^n\|_{X^i} < 2\epsilon$ for $1 \leq i \leq k$. Since ϵ is arbitrary it follows that $G \in \mathcal{H}_t$.

Using the monotone class theorem ((2.3), chapter 2; or the $\pi-\lambda$ theorem) it follows that $\mathcal{H}_t$ contains all bounded predictable processes that vanish on (t,∞). If $K \in \Pi_2(X^i)$ for $1 \leq i \leq k$ and we define $K^n = K1_{[|K|\leq n]}1_{[0,n]}$, then the dominated convergence theorem implies that $\|K^n - K\|_{X^i} \to 0$. Since K^n is bounded and vanishes on (n,∞), another use of the triangle inequality implies that K is a limit of $H^n \in b\Pi_1$. □

Theorem 4.6: $\mathcal{M}^2$ is complete.

Proof: Standard martingale convergence theorems imply that if $X \in \mathcal{M}^2$, then as $t \to \infty$, X_t converges almost surely and in L^2 to a limit X_∞ with $EX_\infty^2 = \sup_t EX_t^2$, and the martingale can be recovered from $X_\infty \in \mathcal{F}_\infty$ by $X_t = E(X_\infty|\mathcal{F}_t)$. Let $\mathcal{F}_\infty = \sigma[\mathcal{F}_t,\ t \geq 0]$. Since $X_\infty = \lim X_t \in \mathcal{F}_\infty$, the observation above shows that $X \to X_\infty$ maps $\mathcal{M}^2$ one-to-one into $L^2(\mathcal{F}_\infty)$.

On the other hand, if $Y \in L^2(\mathcal{F}_\infty)$, then $Y_t = E(Y|\mathcal{F}_t)$ is a martingale with $Y_t \to Y$ as $t \to \infty$, and Jensen's inequality shows that

$$EY_t^2 = E(E(Y|\mathcal{F}_t)^2) \leq E(E(Y^2|\mathcal{F}_t)) = EY^2,$$

so $Y_t \in \mathcal{M}^2$. Combining this with the previous observation shows that $X \to X_\infty$ is an isometry from $\mathcal{M}^2$ onto $L^2(\mathcal{F}_\infty)$ and proves (4.6). □

With (4.5) and (4.6) in hand, we can give:

Definition of the integral for $H \in \Pi_2(X)$**:** To define $H \cdot X$ when X is a bounded continuous martingale and $H \in \Pi_s(X)$, let $H^n \in b\Pi_1$ so that $\|H^n - H\|_X \to 0$. Since $\|H^n - H^m\|_X \to 0$ as $m, n \to \infty$, $(H^n \cdot X)$ is a Cauchy sequence in $\mathcal{M}^2$ and by (4.6) must converge to a limit in $\mathcal{M}^2$, which we define to be the integral $(H \cdot X)$. To see that the limit is independent of the sequence of approximations chosen, suppose $\|H^n - H\|_X \to 0$ and $\|\bar{H}^n - H\| \to 0$ with $(H^n \cdot X) \to Y$ and $(\bar{H}^n \cdot X) \to \bar{Y}$. Form a third approximating sequence by setting $\tilde{H}^n = H^n$ if n is odd, $\tilde{H} = \bar{H}_n$ if n is even. Then $\|\tilde{H}_n - H\|_X \to 0$, so we have $\tilde{H}^n \cdot X \to Z$. Looking at the even and odd subsequences of $\tilde{H}_n$ it follows that $Y = Z = \bar{Y}$, so the limit is unique.

The following theorem is a corollary of the construction:

Theorem (4.3.a): If X is a bounded continuous martingale and $H \in \Pi_2(X)$, then $H \cdot X \in \mathcal{M}^2$ and is continuous.

Proof: The fact that $H \cdot X \in \mathcal{M}_2$ is automatic from the definition. If $H^n \in b\Pi_1$ with $\|H^n - H\|_X \to 0$, then (4.2.a) implies $(H^n \cdot X)_t$ is continuous. Since $\|(H^n \cdot X) - (H \cdot X)\|_2 \to 0$, using Chebychev and the L^2 maximal inequality yields

$$\begin{aligned} P\left(\sup_t |(H^n \cdot X)_t - (H \cdot X)_t| > \epsilon\right) &\leq \epsilon^{-2} E \sup_t |(H^n \cdot X)_t - (H \cdot X)_t|^2 \\ &\leq \epsilon^{-2} 4 \|(H^n \cdot X) - (H \cdot X)\|_2 \to 0. \end{aligned}$$

since this holds for any $\epsilon > 0$ we say "$(H^n \cdot X)_t$ converges uniformly to $(H \cdot X)_t$ in probability". By passing to a subsequence we can have

$$\sup_t |(H^n \cdot X)_t - (H \cdot X)_t| \to 0 \quad \text{a.s.},$$

and it follows that $t \mapsto (H \cdot X)_t$ is continuous. □

Theorem 4.3.b: If X is a bounded continuous martingale, and $H, K \in \Pi_2(X)$, then $H + K \in \Pi_2(X)$ and

$$((H+K)\cdot X)_t = (H\cdot X)_t + (K\cdot X)_t.$$

Proof: The triangle inequality for the norm $\|\cdot\|_X$ implies that $H + K \in \Pi_2(X)$. Let $H^n, K^n \in b\Pi_1$ with $\|H^n - H\|_X \to 0$ and $\|K^n - K\|_X \to 0$. The triangle inequality implies $\|(H^n + K^n) - (H+K)\|_X \to 0$. Then (4.2.b) implies that

$$((H^n+K^n)\cdot X)_t = (H^n\cdot X)_t + (K^n\cdot X)_t.$$

Now let $n \to \infty$ and use the fact that $(G^n \cdot X)_t \to (G \cdot X)_t \in \mathcal{M}^2$ where $G = H, K$, or $H + K$. $\square$

The proofs of the remaining two equalities

$$\begin{aligned} (H\cdot(X+Y))_t &= (H\cdot X)_t + (H\cdot Y)_t \\ \langle H\cdot X, K\cdot Y\rangle &= \int_0^t H_s K_s d\langle X, Y\rangle_s \end{aligned}$$

will be given in the next section.

To establish the results there we will need to extend the isometry property from $b\Pi_1$ to $\Pi_2(X)$. To do this it is useful to recall some basic (undergraduate?) analysis.

2.5: The Kunita - Watanabe Inequality

Theorem 5.1: If X and Y are local martingales and H and K are two measurable processes, then almost surely

$$\int_0^\infty |H_s K_s| d|\langle X, Y\rangle|_s \leq \left(\int_0^\infty H_s^2 d\langle X\rangle_s\right)^{1/2} \left(\int_0^\infty K_s^2 d\langle Y\rangle_s\right)^{1/2}$$

where $d|\langle X, Y\rangle|_s$ stands for dV_s where V_s is the total variation of $r \mapsto \langle X, Y\rangle_r$ on $[0, s]$.

Remark (i): If $X = Y$ and $d\langle X\rangle_s = s$, then $d\langle Y\rangle_s = d|\langle X, Y\rangle|_s = s$ and (5.1) reduces to the Cauchy-Schwarz inequality.

Remark (ii): Notice that H and K are not assumed to be predictable. We assume only that $H(s,\omega)$ and $K(s,\omega)$ are measurable with respect to $\mathcal{R}\times\mathcal{F}$ where $\mathcal{R}$ is the Borel subsets of $\mathbb{R}$.

Proof: Step 1: Note that if $s \le t$,
$\langle X+\lambda Y, X+\lambda Y\rangle_t \ge \langle X+\lambda Y, X+\lambda Y\rangle_s$.
If we let $\langle M,N\rangle_s^t = \langle M,N\rangle_t - \langle M,N\rangle_s$, then

$$\begin{aligned} 0 &\le \langle X+\lambda Y, X+\lambda Y\rangle_t - \langle X+\lambda Y, X+\lambda Y\rangle_s \\ &= \langle X\rangle_s^t - 2\lambda\langle X,Y\rangle_s^t + \lambda^2\langle Y\rangle_s^t \end{aligned}$$

for all s,t, and λ. If we fix s and t and throw away a countable number of sets of measure 0 then with probability one the last inequality will hold for all rational λ. Now a quadratic ax^2+bx+c that is nonnegative at all the rationals and not identically 0 has at most one real root (i.e. $b^2-4ac\le 0$), so it follows that

$$(\langle X,Y\rangle_s^t)^2 \le \langle X\rangle_s^t\langle Y\rangle_s^t.$$

Step 2: Let $0 = t_0 < t_1 < \cdots < t_n$ be an increasing sequence of times, let h_i, k_i, $1 \le i \le n$, be random variables, and define simple measurable processes

$$H(s,\omega) = \sum_{i=1}^{n} h_i(\omega) 1_{(t_{i-1},t_i]}(s), \;\; K(s,\omega) = \sum_{i=1}^{n} k_i(\omega) 1_{(t_{i-1},t_i]}(s).$$

From the definition of the integral, the result of Step 1, and the Cauchy-Schwarz inequality it follows that

$$\begin{aligned}
\left| \int_0^\infty H_s K_s d\langle X,Y\rangle_s \right| &\le \sum_{i=1}^{n} |h_i k_i| |\langle X,Y\rangle_{t_{i-1}}^{t_i}| \\
&\le \sum_{i=1}^{n} |h_i| \left(\langle X,X\rangle_{t_{i-1}}^{t_i} \langle Y,Y\rangle_{t_{i-1}}^{t_i} \rangle \right)^{1/2} |k_i| \\
&\le \left(\sum_{i=1}^{n} h_i^2 \langle X,X\rangle_{t_{i-1}}^{t_i} \right)^{1/2} \left(\sum_{i=1}^{n} k_i^2 \langle Y,Y\rangle_{t_{i-1}}^{t_i} \right)^{1/2},
\end{aligned}$$

proving that for simple measurable processes we have

$$\left|\int_0^\infty H_s K_s d\langle X, Y\rangle_s\right| \le \left(\int_0^\infty H_s^2 d\langle X\rangle_s\right)^{1/2} \left(\int_0^\infty K_s^2 d\langle Y\rangle_s\right)^{1/2}, \quad (5.2)$$

which is (5.1) with the absolute values outside the integral.

Step 3: Let M be a large number and let $T \equiv \inf\{t : \langle X\rangle_t \text{ or } \langle Y\rangle_t > M\}$. By the monotone convergence theorem it suffices to prove (5.1) when $H = K = 0$ for $s \ge T$ and $|H_s|, |K_s| < M$ for $s < T$. Having restricted our attention to $[0, T]$, $\langle X\rangle$ and $\langle Y\rangle$ are finite measures; so using the bounded convergence theorem we see that (5.2) holds for bounded measurable processes. To improve (5.2) to (5.1) (and complete the proof), let J_s be a measurable process taking values in $\{-1, 1\}$ such that (troubles here? $J_s = d\langle X, Y\rangle_s / d|\langle X, Y\rangle_s|$?)

$$\int_0^t |d\langle X, Y\rangle_s| = \int_0^t J_s d\langle X, Y\rangle_s.$$

The process J_s exists since it is the Radon-Nikodym derivative of $\langle X, Y\rangle_s$ with respect to $d|\langle X, Y\rangle_s|$. Now apply (5.2) to $H_s = |H_s|$ and $K_s = J_s|K_s|$. □

The following theorems take care of unfinished business from section 4.

Theorem 5.3: If $H \in \Pi_2(X) \cap \Pi_2(Y)$, then $H \in \Pi_2(X+Y)$ and

$$(H \cdot (X+Y))_t = (H \cdot X)_t + (H \cdot Y)_t.$$

Proof: By Exercise 3.2 we know that

$$\langle X+Y\rangle_t = \langle X\rangle_t + \langle Y\rangle_t + 2\langle X, Y\rangle_t,$$

and the Kunita-Watanabe inequality (5.1) implies that

$$|\langle X, Y\rangle|_t \le (\langle X\rangle_t \langle Y\rangle_t)^{1/2} \le (\langle X\rangle_t + \langle Y\rangle_t)/2$$

since $2ab \le a^2 + b^2$. So it follows that

$$\langle X+Y\rangle_t \le 2(\langle X\rangle_t + \langle Y\rangle_t)$$

(which is reminiscent of the C_r-inequality with $r = 2$). It follows that $H \in \Pi_2(X + Y)$. To prove the formula, note that by (4.5) we can find $H^n \in b\Pi_1$ so that $\|H^n - H\|_Z \to 0$ for $Z = X, Y, X+Y$. (4.2.b) implies that

$$(H^n \cdot (X + Y))_t = (H^n \cdot X)_t + (H^n \cdot Y)_t.$$

Now let $n \to \infty$ and use $(H^n \cdot Z) \to (H \cdot Z)_t$ for $Z = X, Y, X + Y$. □

Theorem 5.4: If X, Y are bounded continuous martingales, $H \in \Pi_2(X)$, $K \in \Pi_2(Y)$, then

$$\langle H \cdot X, K \cdot Y \rangle_t = \int_0^t H_s K_s d\langle X, Y \rangle_s.$$

Proof: By (3.11) it suffices to show that

$$Z_t = (H \cdot X)_t (K \cdot Y)_t - \int_0^t H_s K_s d\langle X, Y \rangle_s \qquad (\dagger)$$

is a martingale. Let H^n and K^n be sequences of elements of $b\Pi_1$ that converge to H and K in $\Pi_2(X)$ and $\Pi_2(Y)$, respectively, and let Z_t^n be the quantity that results when H^n and K^n replace H and K in (†). By (4.2.c), Z_t^n is a martingale. By (3.6), we can complete the proof by showing $Z_s^n \to Z_s$ in L^1. The triangle inequality and (4.3.b) imply that

$$
\begin{aligned}
&E \sup_t |(H^n \cdot X)_t (K^n \cdot Y)_t - (H \cdot X)_t (K \cdot Y)_t| \\
&\quad \le \; E \sup_t |((H^n - H) \cdot X)_t (K^n \cdot Y)_t| \\
&\qquad + \; E \sup_t |(H \cdot X)_t ((K^n - K) \cdot Y)_t|.
\end{aligned}
$$

To bound the first term, we use: (i) the sup of the product is smaller than the product of the sup's; (ii) the Cauchy-Schwarz inequality, (iii) the L^2 maximal inequality, and (iv) the isometry property in Exercise 4.5:

$$
\begin{aligned}
&\leq E\{\sup_t |((H^n - H)\cdot X)_t| \sup_t |(K^n \cdot Y)_t|\} \\
&\leq \{E(\sup_t |((H^n - H)\cdot X_t|^2) E(\sup_t |(K^n \cdot Y)_t|^2)\}^{1/2} \\
&\leq 4\|(H^n - H)\cdot X\|_2 \|K^n \cdot Y\| \\
&= 4\|H^n - H\|_X \|K^n\|_Y \to 0 \quad \text{as} \quad n \to \infty
\end{aligned}
$$

since $\|H^n - H\|_X \to 0$ and $\|K^n Y\|_Y \to \|K\|_Y < \infty$. A similar estimate shows

$$
E\left(\sup_t |(H\cdot X)_t((K^n - K)\cdot Y)_t|\right) \to 0,
$$

so we have shown that $(H^n \cdot X)_s(K^n \cdot Y)_s \to (H\cdot X)_s(K\cdot Y)_s$ in L^1.

To estimate the second term in $Z_t^n - Z_t$, we again begin by putting the absolute values inside, replacing the measure by its variation, and then using the triangle inequality:

$$\begin{aligned}
&\left|\int_0^t H_s^n K_s^n d\langle X,Y\rangle_s - \int_0^t H_s K_s d\langle X,Y\rangle_s\right| \\
&\le \int_0^t |H_s^n K_s^n - H_s K_s| d|\langle X,Y\rangle_s \\
&\le \int_0^t |H_s^n - H_s||K_s^n| d|\langle X,Y\rangle|_s + \int_0^t |H_s||K_s^n - K_s| d\langle X,Y\rangle|_s.
\end{aligned}$$

Using the Kunita-Watanabe inequality (5.1), the first term is

$$\le \left(\int_0^t |H_s^n - H_s|^2 d\langle X\rangle_s\right)^{1/2} \left(\int_0^t |K_s^n|^2 d\langle Y\rangle_s\right)^{1/2}.$$

Taking expected values and using the Cauchy - Schwarz inequality we find

$$E\int_0^t |H_s^n - H_s||K_s^n| d|\langle X,Y\rangle|_s$$
$$\leq \left(E\int_0^t |H_s^n - H_s|^2 d\langle X\rangle_s\right)^{1/2} \left(E\int_0^t |K_s^n|^2 d\langle Y\rangle_s\right)^{1/2} \to 0$$

as $n \to \infty$, since $\|H^n - H\|_X \to 0$ and $\|K^n\|_Y \to \|K\|_Y < \infty$. Combining this with a similar bound on $\int_0^t |H_s||K_s^n - K_s| d|\langle X,Y\rangle|_s$ it follows that

$$\int_0^t H_s^n K_s^n d\langle X,Y\rangle_s \to \int_0^t H_s K_s d\langle X,Y\rangle_s$$

in L^1. We have now shown $Z_s^n \to Z_s$ in L^1, and the desired result follows from (3.6). □

2.6: Integration w.r.t. Local Martingales

Now we will extend our integral so that the integrators can be continuous local martingales and so that the integrands are in

$$\Pi_3(X) = \left\{ H : \int_0^t H_s^2 d\langle X \rangle_s < \infty \text{ a.s. for all } t \geq 0 \right\}.$$

In fact this is the largest possible class of integrands. To extend the integral from bounded martingales we begin by proving the following obvious fact:

Theorem 6.1: Suppose that X is a bounded continuous martingale, $H, K \in \Pi_2(X)$ and $H_s = K_s$ for $s \leq T$ where T is a stopping time. Then $(H \cdot X)_s = (K \cdot X)_s$ for $s \leq T$.

Proof: □

To extend the integral, let S_n be a sequence of stopping times with

$$S_n \leq T_n \equiv \inf \left\{ t : |X_t| > n \text{ or } \int_0^t H_s^2 d\langle X \rangle_s > n \right\}.$$

and $S_n \nearrow \infty$. Let $H^n_s = H_s 1_{(s<S_n)}$, and observe that if $m < n$, (6.1) implies that $(H^m \cdot X)_t = (H^n \cdot X)_t$ for $t \le S_m$, so we can define $H \cdot X$ by setting $(H^n \cdot X)_t$ for $t \le S_n$.

To complete the definition we have to show that if R_n and S_n are two sequences of stopping times $\le T_n$ with $R_n \nearrow \infty$ and $S_n \nearrow \infty$, then we end up with the same $(H \cdot X)_t$. Let H^T_s be the stopped version of the process defined at the beginning of Section 2.2, let $Q_n = R_n \wedge S_n$, and note that (6.1) implies

$$(H^{R_n} \cdot X)_s = (H^{S_n} \cdot X)_s \quad \text{for } s \le Q_n.$$

Since $Q_n \nearrow \infty$, it follows that $(H \cdot X)_t$ is independent of the sequence of stopping times chosen. The uniqueness result and (6.1) imply:

Theorem 6.2: If X is a continuous local martingale and $H \in \Pi_3(X)$, then

$$H^T \cdot X = (H \cdot X)^T = H \cdot X^T = H^T \cdot X^T.$$

In words, if we set the integrand $= 0$ after time T, or stop the martingale at time T or do both, then this just stops the integral at T. s

The next job is to extend the key results (a), (b), (c) to integrands in $\Pi_3(X)$.

Theorem (6.3): If X is a continuous local martingale and $H \in \Pi_3(X)$, then $(H \cdot X)_t$ is a continuous local martingale.

Theorem (6.4): Let X and Y be continuous local martingales. If $H, K \in \Pi_3(X)$ the $H + K \in \Pi_3(X)$ and

$$((H + K) \cdot X)_t = (H \cdot X)_t + (K \cdot X)_t.$$

If $H \in \Pi_3(X) \cap \Pi_3(Y)$, then $H \in \Pi_3(X + Y)$ and

$$(H \cdot (X + Y))_t = (H \cdot X)_t + (H \cdot Y)_t.$$

Proof: To prove the first formula we note that the triangle inequality for the norm $\|\cdot\|_X$ implies $H+K \in \Pi_2(X)$. Stopping at

$$T_n = \inf\left\{t:\ |X_t|, \int_0^t H_s^2 d\langle X\rangle_s, \text{ or } \int_0^t K_s^2 d\langle Y\rangle_s \ > n\ \right\}$$

reduces the result to (4.3.b).

For the second formula, note that the argument in (5.3) shows that $H \in \Pi_3(X+Y)$ and by stopping we can reduce the result to (5.3). $\square$

Given the last two proofs, the following improvement of (5.4) should be clear:

Theorem 6.5: If X, Y are continuous local martingales, $H \in \Pi_3(X)$ and $K \in \Pi_3(X)$, then

$$\langle H \cdot X, K \cdot Y \rangle_t = \int_0^t H_s K_s d\langle X, Y \rangle_s.$$

This generalizes immediately to sums of stochastic integrals:

Theorem (6.6): Let $X = \sum_{i=1}^m H^i \cdot X_i$ and $Y = \sum_{j=1}^n K^j \cdot Y^j$ where the X^i and Y^j are continuous local martingales. If $H^i \in \Pi_3(X^i)$ and $K^j \in \Pi_3(Y^j)$, then

$$\langle X, Y \rangle_t = \sum_{i,j} \int_0^t H_s^i K_s^j d\langle X^i, Y^j \rangle_s \ .$$

2.7: Change of Variables, Itô's Formula