Math/Stat 523, Spring 2020

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Lecture 10

Friday May 1, Monday May 4

Outline

- 0: Discrete time martingale theory: a reminder
- 1: Integrands and integrators (Sections 2.1 and 2.2)
- 2: Variance-covariance processes ; integration w.r.t. bounded martingales
  Sections 2.3 and 2.4.
- 3: Kunita-Watanabe inequality; integration w.r.t. local martingales.
  Sections 2.5 and 2.6.
- 4: Ito's formula ; integration w.r.t. semimartingales Sections 2.7 and 2.8

0. Discrete time martingale theory

2.1: Integrands: predictable processes

- Recall the martingale transform in the setting of discrete-time martingales:
  - ▷ Suppose that $\{X_n : \ n \geq 0\}$ is a martingale and $\{H_n : \ n \geq 1\}$ is any process:
  - ▷ The martingale transform $(H \cdot X)_n$ is defined by

$$(H \cdot X)_n \equiv \sum_{m=1}^{n} H_m (X_m - X_{m-1}).$$

  - ▷ If $\{H_n\}$ is predictable (i.e. $H_n \in \mathcal{F}_{n-1}$ for each n, then $(H \cdot X)_n$ is a martingale.

- **Example:** Let $(\Omega, \mathcal{A}, P)$ be a probability space, and suppose that $T \sim U(0,1)$, $\xi \sim$ Rademacher, are both defined on $(\Omega, \mathcal{A}, P)$, and are independent. Let $X_t \equiv \xi 1_{[T \le t]}$ and $\mathcal{F}_t \equiv \sigma[X_s : \ s \le t]$. Then $\{X_t, \mathcal{F}_t\}_{t \in [0,1]}$ is a martingale. But

$$I_t \equiv \int_0^t X_s dX_s = X_T \cdot \xi = \xi^2 = 1,$$

so $I_0 = 0$, but $I_1 = 1$ and $\{I_t : t \ge 0\}$ is not a martingale.

- To preserve the martingale property we will need to restrict to left-continuous integrands.

- For $a < b$, $C \in \mathcal{F}_a$, let $H(s,w) = C(w)1_{(a,b]}(s)$.
- Let $\mathcal{P} \equiv \Pi \equiv \sigma[\mathcal{C}]$ where

$$\mathcal{C} = \{(a,b] \times A : \ A \in \mathcal{F}_a, \ 0 \le a < b < \infty\}$$

- Let $\widetilde{\mathcal{P}} \equiv \sigma[H : \ H$ is left-continuous and adapted$]$.

Theorem: $\mathcal{P} = \widetilde{\mathcal{P}} \equiv$ the predictable $\sigma -$ field .

- Optional $\sigma-$field $\mathcal{O} \equiv \sigma[H$ adapted, right-cont, with left limits$]$.
- Progressive $\sigma-$field $\mathcal{M}$:

$$\equiv \sigma[H : H(s,\omega)\text{from } [0,t] \times \Omega \text{ is } \mathcal{B}_t \times \mathcal{F}_t \text{ measurable}]$$

2.2: Integrators: Continuous local martingales

- $s \mapsto X_s(\omega)$ is continuous.
- **Definition:** $\{X_t : t \geq 0\}$ is a local martingale (w.r.t. $\mathcal{F}_t$) if there are stopping times $T_n \nearrow \infty$ so that $X_t^{T_n}$ is a martingale (w.r.t. $\{\mathcal{F}_{t\wedge T_n} : t \geq 0\}$. The stopping times T_n are said to reduce X. Here for any non-negative rv T and any process Y_t we define

$$Y_t^T = \begin{cases} Y_{T\wedge t} & \text{on } \{T > 0\} \\ 0 & \text{on } \{T = 0\}. \end{cases}$$

- Three reasons for introducing local martingales!
  Durrett page 38!

Theorem 2.2: $\{X_{\gamma(t)}, \mathcal{F}_{\gamma(t)} : t \geq 0\}$ is a martingale.

Theorem 2.3: (Optional sampling theorem) Let X be a continuous local martingale. If $S \leq T$ are stopping times and $X_{T\wedge t}$ is a uniformly integrable martingale, then $E(X_T|\mathcal{F}_S) = X_S$ almost surely.

Theorem 2.4: If X is a continuous local martingale we can always take the sequence which reduces X to be $T_n \equiv \inf\{t : |X_t| > n\}$ or any other sequence $T_n' \leq T_n$ that has $T_n' \nearrow \infty$ as $n \to \infty$.

Theorem 2.5: If X_t is a local submartingale and

$$E\left(\sup_{0\leq s\leq t} |X_s|\right) < \infty, \tag{1}$$

for each t, then X_t is a sub-martingale.

Proof: The hypothesis (1) implies that $E|X_t| < \infty$ for each t, so it remains just to prove $E(X_t|\mathcal{F}_s) \geq X_s$ a.s. To accomplish this, note that if $\{T_n\}$ is a sequence of stopping times that reduces X, it follows that

$$E\left(X_t^{T_n}|\mathcal{F}_s^{T_n}\right) \geq X_s^{T_n}.$$

Let $n \to \infty$ and apply the dominated convergence theorem for conditional expectations. (Hunt's lemma ?) □

Multiplying by -1 shows that the last result holds for super-martingales, so it is also true for martingales (by the results for both sub- and super-martingales).

Corollary: A bounded local martingale is a martingale.

Here *bounded* means that there is a constant M so that with probability 1, $|X_t| \leq M$ for all $t \geq 0$.

Theorem: Let X_t be a local martingale on $[0, \tau)$. If

$$E\left(\sup_{0 \le s < \tau} |X_s|\right) < \infty$$

then $X_\tau = \lim_{t \nearrow \tau} X_t$ exists a.s. and $EX_0 = EX_\tau$.

Proof: Let γ be the time scale of (2.2) and let $Y_t = X_\gamma(t)$. Y_t is a martingale. The upcrossing inequality and the martingale convergence theorem apply and yield that conclusion that $Y_\infty = \lim_{t \nearrow \infty} Y_t$ exists a.s. and $X_\tau = \lim_{t \nearrow \tau} X_t$ exists a.s. To see the equality of expectations note that $EY_0 = EY_t$ then use the a.s. convergence and the dominated convergence theorem to conclude $EX_0 = EY_0 = EY_\infty = EX_\tau$. $\square$

2.3 Variance and Co-variance processes

Theorem 3.1 If X_t is a continuous local martingale, then we define the (predictable) variance process $\langle X\rangle_t$ to be the unique continuous predictable increasing process A_t that has $A_0 = 0$ and makes $X_t^2 - A_t$ a local martingale.

The proof of this will be carried out in several stages. First we recall the corresponding result in discrete time:

Theorem 3.2 Suppose that $\{X_n, \mathcal{F}_n\}$ is a martingale with $EX_n^2 < \infty$ for all n. Then there is a unique predictable process $\{A_n\}_{n\geq 0}$ with $A_0 = 0$ so that $\{X_n^2 - A_n\}$ is a martingale. Furthermore, A_n in increasing.

In fact, we see that

$$A_n = \sum_{k=1}^{n} \{E(X_k^2|\mathcal{F}_{k-1}) - X_{k-1}^2\}.$$

The uniqueness part of Theorem 3.2 boils down to "any predictable discrete time martingale is constant". Since Brownian motion is a predictable martingale, this statement fails in continuous time. Thus we impose the assumption of "locally of bounded variation".

Theorem 3.3 A continuous local martingale X_t that is predictable and locally of bounded variation is constant (in time).

Proof: By subtracting X_0 we may suppose that $X_0 = 0$ and prove that X is identically 0. Let V_t be the variation of X on $[0, t]$. Then the result holds since for any process X that is continuous and locally of bounded variation then $t \mapsto V_t$ is continuous. $\square$

To get underway with the proof of Theorem 3.1, define $S \equiv \inf\{s : V_s \geq K\}$. Since $t \leq S$ implies $|X_t| \leq K$ (2.4) implies that the stopped process $M_t = X(t \wedge S)$ is a bounded martingale. Now if $s < t$ using well known properties of conditional expectation, we have

$$\begin{aligned} E((M_t - M_s)^2|\mathcal{F}_s) &= E(M_t^2|\mathcal{F}_s) - 2M_s E(M_t|\mathcal{F}_s) + M_s^2 \qquad (3.4) \\ &= E(M_t^2|\mathcal{F}_s) - M_s^2 = E(M_t^2 - M_s^2|\mathcal{F}_s), \end{aligned}$$

We will refer to this as the "orthogonality of martingale increments" since the key to its proof is

$$E(M_s(M_t - M_s)|\mathcal{F}_s) = 0 \quad \text{and hence} \quad E\{M_s(M_t - M_s)\} = 0.$$

Let $0 = t_0 < t_1 < \cdots < t_n = t$ be a subdivision of $[0, t]$. Then by the orthogonality of martingale increments, the inequality $\sum_i a_i^2 \leq \sup_j |a_j| (\sum_i |a_i|)$, and $V_{t \wedge S} \leq K$ imply

$$
\begin{aligned}
EM_t^2 &= E\left(\sum_{m=1}^{n} M_{t_m}^2 - M_{t_{m-1}}^2\right) \\
&= E\sum_{m=1}^{n} (M_{t_m} - M_{t_{m-1}})^2 \\
&\le E\left(V_{t\wedge S} \sup_m |M_{t_m} - M_{t_{m-1}}|\right) \\
&\le KE\sup_m |M_{t_m} - M_{t_{m-1}}|.
\end{aligned}
$$

If we take a sequence of partitions $\Delta_n = \{0 = t_0^n < t_1^n < \cdots < t_{k(n)}^n = t\}$ in which the mesh $|\Delta_n| = \sup_m |t_m^n - t_{m-1}^n| \to 0$, then continuity implies $\sup_m |M_{t_m^n} - M_{t_{m-1}}^n| \to 0$. This shows $EM_t^2 = 0$ so $M_t^2 = 0$ a.s. Since t is arbitrary, with probability one we have $M_t = 0$ for all rational t. The claimed result now follows from continuity. □

Proof of uniqueness: With Theorem 3.3 in hand, we can establish the uniqueness part of Theorem 3.1. Suppose that A_t and A'_t are two processes with the desired properties. Then $A_t - A'_t$ is a continuous local martingale that is locally of bounded variation and hence must be constant. Since $A_0 - A'_0 = 0$, it follows that $A_t - A'_t = 0$ for all t. □

Proof of existence in (3.1), X_t a bounded martingale: We are now going to prove the special case of the Doob-Meyer decomposition that we need.

Given a partition $\Delta = \{0 = t_0 < t_1 < t_2 \cdots\}$ with $\lim_n t_n = \infty$, let $k(t) = \sup_{k:t_k<t}$ be the index of the last point before time t. (Note that $k(t)$ is a number, not a random variable.) Then define for a process X, an approximate quadratic variation by

$$Q_t^{\Delta}(X) \equiv \sum_{k=1}^{k(t)} (X_{t_k} - X_{t_{k-1}})^2 + (X_t - X_{t_{k(t)}})^2.$$

Lemma (a): If X_t is a bounded continuous martingale, then $X_t^2 - Q_t^{\Delta}(X)$ is a martingale.

Proof: First note that reasoning as in the proof of (3.4) we have, if $r < s < t$,

$$
\begin{aligned}
&E\left((X_t - X_r)^2 | \mathcal{F}_s\right) \\
&\quad = \; E\left((X_t - X_s)^2 | \mathcal{F}_s\right) - 2(X_s - X_r)^2 E(X_t - X_s | \mathcal{F}_s) + (X_s - X_r)^2 \\
&\quad = \; E\left((X_t - X_s)^2 | \mathcal{F}_s\right) + (X_s - X_r)^2
\end{aligned}
$$

since $E(X_t - X_s | \mathcal{F}_s) = 0$.

Writing $Q_t^{\Delta}(X) = Q_s^{\Delta}(X) + (Q_t^{\Delta}(X) - Q_s^{\Delta}(X))$ and computing the difference term we find

$$
\begin{aligned}
Q_t^{\triangle}(X) - Q_s^{\triangle}(X) &= \sum_{k=1}^{k(t)} (X_{t_k} - X_{t_{k-1}})^2 + (X_t - X_{t_{k(t)}})^2 \\
&\quad - \sum_{k=1}^{k(s)} (X_{t_k} - X_{t_{k-1}})^2 + (X_s - X_{t_{k(s)}})^2 \\
&= (X_{t_{k(s)+1}} - X_{t_{k(s)}})^2 - (X_s - X_{(t_{k(s)}})^2 \\
&\quad + \sum_{k=k(s)+2}^{k(t)} (X_{t_k} - X_{t_{k-1}})^2 + (X_t - X_{t_{k(t)}})^2.
\end{aligned}
$$

Now we take the conditional expectation with respect to $\mathcal{F}_s$ and use the first formula with $r = t_{k(s)}$ and $t = t_{k(s)+1}$. This yields, upon defining u_n for $k(s)-1 \leq n \leq k(t)+1$ by setting $u_{k(s)-1} = s$, $u_i = t_i$ for $k(s) \leq i \leq k(t)$, and $u_{k(t)+1} = t$,

$$\begin{aligned}
&E(X_t^2 - Q_t^{\Delta}(X)|\mathcal{F}_s) \\
&= E(X_t^2|\mathcal{F}_s) - Q_s^{\Delta}(X) - E\left(\sum_{i=k(s)+1}^{k(t)+1} (X_{u_i} - X_{u_{i-1}})^2 \middle| \mathcal{F}_s\right) \\
&= E(X_t^2|\mathcal{F}_s) - Q_s^{\Delta}(X) - E\left(\sum_{i=k(s)+1}^{k(t)+1} X_{u_i}^2 - X_{u_{i-1}}^2 \middle| \mathcal{F}_s\right) \\
&= X_s^2 - Q_s^{\Delta}(X)
\end{aligned}$$

where the second equality follows from (3.4). □

Looking at (a) and noticing that $Q_t^{\Delta}(X)$ is increasing except for the last term, it becomes clear that one reasonable strategy for constructing the process A_t is to take a sequence of partitions Δ_n with mesh converging to 0 and proving the the limit exists. Here is an identity which will help: by (a), taking the conditional expectation, and using (3.4) yields

$$\begin{aligned} E(Q_t^{\Delta}(X) - Q_s^{\Delta}(X)|\mathcal{F}_s) &= E(X_t^2 - X_s^2|\mathcal{F}_s) \\ &= E((X_t - X_s)^2|\mathcal{F}_s). \end{aligned}$$

The following lemma contains the crux of the argument:

Lemma (c): Let X_t be a bounded continuous martingale. Fix $r > 0$ and let Δ_n be a sequence of partitions $0 < t_0^n < t_1^n < \cdots < t_{k_n}^n = r$ of $[0, r]$ with mesh $|\Delta_n| = \sup_k |t_k^n - t_{k-1}^n| \to 0$. Then $Q_r^{\Delta_n}(X)$ converges to a limit in L_2.

Proof: If Δ and Δ' are two partitions, we call $\Delta\Delta'$ the partition obtained by taking all the points in Δ and in Δ' If we apply (a) twice and differences we see that $Y_t = Q_t^{\Delta}(X) - Q_t^{\Delta'}(X)$ is a martingale. By definition $Y_t^2 - Q_t^{\Delta\Delta'}(Y)$ is a martingale, so

$$E(Q_r^{\Delta}(X) - Q_r^{\Delta'}(X))^2 = E(Y^r) = E(Q_r^{\Delta\Delta'}(Y).$$

We now drop the argument from Q_r^{Δ} when it is X and drop the r when we want to refer to the process $t \mapsto Q_t^{\Delta}$. Since

$$(a+b)^2 \le 2a^2 + 2b^2 \quad \text{for any real numbers } a \text{ and } b,$$

(since $2a^2 + 2b^2 - (a+b)^2 = (a-b)^2 \ge 0$), it follows that

$$Q_r^{\Delta\Delta'}(Y) \le 2Q_r^{\Delta\Delta'}(Q^{\Delta}) + 2Q_r^{\Delta\Delta'}(Q^{\Delta'}).$$

Combininb the last two results about Q we see that it is enough to prove that

$$\text{if} \quad |\Delta| + |\Delta'| \to 0, \quad \text{then} \quad EQ_r^{\Delta\Delta'}(Q^{\Delta}) \to 0.$$

To do this, let $s_k \in \Delta\Delta'$ and $t_j \in \Delta$ so that $t_j \leq s_k < s_{k+1} \leq t_{j_1}$. Recalling the definition of $Q_s^{\Delta}(X)$ and doing some algebra we see that

$$\begin{aligned} Q_{s_{k+1}}^{\Delta} - Q_{s_k}^{\Delta} &= (X_{s_{k+1}} - X_{t_j})^2 - (X_{s_k} - X_{t_j})^2 \\ &= ((X_{s_{k+1}} - X_{s_k})^2 + 2(X_{s_{k+1}} - X_{s_k})(X_{s_k} - X_{t_j}) \\ &= (X_{s_{k+1}} - X_{s_k})(X_{s_{k+1}} + X_{s_k} - 2X_{t_j}). \end{aligned}$$

Summing the squares yields

$$Q_r^{\Delta\Delta'}(Q^{\Delta}) \leq Q_r^{\Delta\Delta'}(X) \sup_k (X_{s_{k+1}} + X_{s_k} - 2X_{t_{j(k)}})$$

where $j(k) = \sup\{j : t_j \leq s_k\}$. By the Cauchy-Schwarz inequality

$$E(Q_r^{\Delta\Delta'}(Q^{\Delta})) \leq \{EQ_r^{\Delta\Delta'}(X)^2\}^{1/2} \left\{E \sup_k (X_{s_{k+1}} + X_{s_k} - 2X_{t_{j(k)}})^4\right\}^{1/2}.$$

When $|\Delta| + |\Delta'| \to 0$ the second factor converges to 0 since X is bounded and continuous, so it suffices to prove:

$$\text{If } |X_t| \le M \quad \text{for all } t \text{ then } EQ_r^{\Delta\Delta'}(X)^2 \le 12M^4. \qquad (e)$$

Proof of (e): If $\Gamma = \Delta\Delta'$ is the partition $0 = s_0 < s_1 < \cdots < s_n = r$, then

$$\begin{aligned} Q_r^\Gamma(X)^2 &= \left(\sum_{m=1}^{n}(X_{s_m} - X_{s_{m-1}})^2\right)^2 \\ &= \sum_{m=1}^{n}(X_{s_m} - X_{s_{m-1}})^4 \qquad (e.1) \\ &\quad + 2\sum_{m=1}^{n-1}(X_{s_m} - X_{s_{m-1}})^2(Q_r^\Gamma(X) - Q_{s_m}^\Gamma(X)\,. \end{aligned}$$

To bound the first term on the right side, note that if $|X_t| \le M$ for all t, then some algebra and (3.4) imply that

$$\begin{aligned} E\sum_{m=1}^{n}(X_{s_m}-X_{s_{m-1}})^4 &\le (2M)^2 E\sum_{m=1}^{n}(X_{s_m}-X_{s_{m-1}})^2 \\ &= 4M^2 E\sum_{m=1}^{n} X_{s_m}^2 - X_{s_{m-1}}^2 \qquad (e.2) \\ &\le 4M^2 E X_r^2 \le 4M^4. \end{aligned}$$

To bound the second term we note that $(X_{s_m}-X_{s_{m-1}})^2 \in \mathcal{F}_{s_m}$, then use (b) and $|X_r - X_{s_m}| \le 2M$ to get

$$\begin{aligned} &E((X_{s_m}-X_{s_{m-1}})^2 \left\{Q_r^\Gamma(X) - Q_{s_m}^\Gamma(X)\right\} | \mathcal{F}_{s_m}) \\ &= (X_{s_m}-X_{s_{m-1}})^2 E((X_r - X_{s_m})^2 | \mathcal{F}_{s_m}) \qquad (e.3) \\ &\le (2M)^2 (X_{s_m}-X_{s_{m-1}})^2. \end{aligned}$$

Taking the expected value in (e.3), summing over m, and using (3.4) as before, we fine that

$$\begin{aligned} E \sum_{m=1}^{n-1} & (X_{s_m} - X_{s_{m-1}})^2 (Q_r^\Gamma(X) - Q_{s_m}^\Gamma(X)) \\ \leq & \; 4M^2 E \sum_{m=1}^{n} X_{s_m}^2 - X_{s_{m-1}} \qquad (e.4) \\ \leq & \; 4M^2 E X_r^2 \leq 4M^4 \end{aligned}$$

Plugging this bound and (e.2) into (e.1) proves (e) which completes the proof of (d) and hence of (c).

Now we put the pieces together. Let Δ_n be the partition with points $k2^{-n}r$ with $0 \leq k \leq 2^n$. Since $Q_t^{\Delta_m} - Q_t^{\Delta_n}$ is a martingale (by (a)), using the L^2 maximal inequality (see e.g. (4.3) in Durrett (1995)) gives

$$E\left(\sup_{t\le r}|Q_t^{\Delta_m}-Q_t^{\Delta_n}|^2\right)\le 4E|Q_r^{\Delta_m}-Q_r^{\Delta_n}|^2.$$

From this and (c) it follows that:

(g) There is a subsequence so that $Q_t^{\Delta_{n(k)}} \to$ a limit A_t uniformly on $[0,r]$.

Proof: Since $Q_r^{\Delta_n}$ converges in L^2, it is a Cauchy sequence in L^2 and we can pick an increasing sequence $n(k)$ so that for $m \ge n(k)$

$$E|Q_r^{\Delta_m}-Q_r^{\Delta_n}|^2 \le 2^{-k}.$$

Using (f) and Chebychev's inequality it follows that

$$P\left(\sup_{t\le r}|Q_t^{\Delta_{n(k+1)}}-Q_t^{\Delta_{n(k)}}|>1/k^2\right)\le k^4 2^{-k}.$$

Since the right side is summable, the Borel-Cantelli lemma yields the claimed convergence. □

By taking subsequences of subsequences and using a diagonal argument we can show that uniform convergence holds on $[0, N]$ for all N. Since each Q_t^{Δ} is continuous, A_t is also continuous. Because the last term $(X_t - X_{k(t)})^2$, Q_t^{Δ} is not increasing. However, if $m \geq n$, then $k \mapsto Q_{k2^{-n}r}^{\Delta_m}$ is increasing, so $k \mapsto A_{k2^{-n}r}$ is increasing. Since n is arbitrary and $t \mapsto A_t$ is continuous it follows that $t \mapsto A_t$ is increasing.

The last issue for the case in which X is a bounded martingale is to check that $M_t^2 - A_t$ is a martingale. To show this we rely on the following lemma (with $p = 2$).

Lemma (3.6): Suppose that for each n, Z_t^n is a martingale w.r.t. $\mathcal{F}_t$, and that for each t, $Z_t^n \to Z_t$ in L^p where $p \geq 1$. Then Z_t is a martingale.

Proof: Since Z_t^n is a martingale, if $s < t$ then $E(Z_t^n|\mathcal{F}_s) = Z_s^n$. The right side converges to Z_s in L_p. To see that the left side converges to Z_s, note that by linearity of conditional expectation followed by Jensen's inequality it follows that

$$\begin{aligned} E|E(Z_t^n|\mathcal{F}_s) - E(Z_t|\mathcal{F}_s)|^p &= E|E(Z_t^n - Z_t|\mathcal{F}_s)|^p \\ &\le EE(|Z_t^n - Z_t|^p|\mathcal{F}_s) = E|Z_t^n - Z_t|^p \to 0. \end{aligned}$$

Taking limits in L^p it follows that $E(Z_t|\mathcal{F}_s) = Z_s$; i.e. Z_t is a martingale. □

Proof of existence in (3.1) when X is a local martingale The first step in extending the result to local martingales is to prove a result about the quadratic variation of a stopped martingale. Recalling the definition of Y_t^T given at the beginning of Section 2.2, the next result should be obvious: the quadratic variation does not increase after process stops.

Lemma (3.7): Let X be a bounded martingale, and let T be a stopping time. Then $\langle X^T \rangle = \langle X \rangle^T$.

Proof: By the optional stopping theorem $(X^T)^2 - \langle X \rangle^T$ is a local martingale, so the result follows from uniqueness. $\square$

Let $\{T_n\}$ be a sequence of stopping times increasing to ∞ so that

$$Y^n = X^{T_n} \cdot 1_{(T_n > 0)} \quad \text{is a bounded martingale.}$$

By the previous result there is a unique continuous predictable increasing process A^n so that $(Y_t^n)^2 - A_t^n$ is a martingale. By (3.7) $A_t^n = A_t^{n+1}$ for $t \leq T_n$, so we can unambiguously define $\langle X \rangle_t = A_t^n$ for $t \leq T_n$. Clearly $\langle X \rangle_t$ is continuous, predictable, and increasing. The definition implies $X^2_{T_n \wedge t} \cdot 1_{(T_n > 0)} - \langle X \rangle_{T_n \wedge t}$ is a martingale, so $X_t^2 - \langle X \rangle_t$ is a local martingale. $\square$

Thus the existence proof is complete

Here is an extension of (c) that will prove to be useful.

Theorem (3.8): Let X_t be a continuous local martingale. For every t and sequence of subdivisions Δ_n of $[0,t]$ with mesh $|\Delta_n| \to 0$ we have

$$\sup_{s \le t} |Q_s^{\Delta_n}(X) - \langle X \rangle_s| \to_p 0.$$

Proof: Let $\delta, \epsilon > 0$. We can find a stopping time S so that X_t^S is a bounded martingale and $P(S \le t) \le \delta$. It is clear that $Q^{\Delta}(X)$ and $Q^{\Delta}(X^S)$ coincide on $[0,S]$. From the definition and (3.7) it follows that $\langle X \rangle$ and $\langle X^S \rangle$ are equal on $[0,S]$. So we have

$$P\left(\sup_{s \le t} |Q_s^{\Delta}(X) - \langle X \rangle_s| > \epsilon\right) \le \delta + P\left(\sup_{s \le t} |Q_s^{\Delta}(X^S) - \langle X^S \rangle_s| > \epsilon\right).$$

Since X^S is a bounded martingale (c) implies that the last term converges to 0 as $|\Delta| \to 0$. $\square$

Definition (3.9): (Polarization: the predictable covariation process) If X and Y are two continuous local martingales, we let

$$\langle X, Y \rangle_t = \frac{1}{4} (\langle X + Y \rangle_t - \langle X - Y \rangle_t)$$

Remark: If X and Y are random variable with mean zero,

$$\begin{aligned} Cov(X, Y) &= E(XY) = \frac{1}{4}\left(E(X+Y)^2 - E(X-Y)^2\right) \\ &= \frac{1}{4}(Var(X+Y) - Var(X-Y)), \end{aligned}$$

so it is natural to call $\langle X, Y \rangle_t$ the predictable covariation process. (I would prefer to avoid calling it the **covariance** of X and Y (as in Durrett).

Given the definitions of $\langle X\rangle_t$ and $\langle X,Y\rangle_t$, it is not too surprising to find that there is a parallel approximating process for the covariation process: For a given partition $\Delta = \{0 = t_0 < t_1 < t_2 < \cdots\}$ with $\lim_n t_n = \infty$ we let $k(t) = \sup\{k : t_k < t\}$ and define

$$Q_t(X,Y) = \sum_{k=1}^{k(t)} (X_{t_k} - X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}}) + (X_t - X_{t_{k(t)}})(Y_t - Y_{t_{k(t)}}).$$

Theorem 3.10: Let X and Y be continuous local martingales. For every t and sequence of partitions Δ_n of $[0,t]$ with mesh $|\Delta_n| \to 0$ we have

$$\sup_{s\le t} |Q_s^{\Delta_n}(X,Y) - \langle X,Y\rangle_s| \to_p 0.$$

Proof: Since

$$Q_s^{\Delta_n}(X,Y) = \frac{1}{4}\Big(Q_s^{\Delta_n}(X+Y) - Q_s^{\Delta_n}(X-Y)\Big)$$

this follows immediately from the definition of $\langle X,Y\rangle$ and (3.8). $\square$

Finally here is another justification for the notation and terminology:

Theorem 3.11: Suppose X_t and Y_t are continuous local martingales. Then $\langle X,Y\rangle_t$ is the unique continuous predictable process A_t that is locally of bounded variation, has $A_0 = 0$, and makes $X_tY_t - A_t$ a local martingale.

Proof: From the definition it is easy to see that

$$X_tY_t - \langle X, Y\rangle_t$$
$$= \frac{1}{4}[(X_t + Y_t)^2 - \langle X + Y\rangle_t - \{(X_t - Y_t)^2 - \langle X - Y\rangle_t\}]$$

is a local martingale. To prove uniqueness, observe that if A_t and A'_t are two processes with the desired properties, then $A_t - A'_t = (X_tY_t - A'_t) - (X_tY_t - A_t)$ is a continuous local martingale that is locally of bounded variation and hence $\equiv 0$ by (3.3). □