Math/Stat 523, Spr 2020

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Lecture 1

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Outline

- 1: The strong Markov property
- 2: Example 1
- 3: Example 2

Theorem 5.1 (Strong Markov) Suppose that

$$X : (\Omega, \mathcal{A}, P) \to (D_{[0,\infty)}, \mathcal{D}_{[0,\infty)})$$

are adapted to right continuous $\mathcal{A}_t$'s. Suppose that

- $X(0) = 0$
- X has stationary and independent increments.
- $X(t+s) - X(t)$ is independent of $\mathcal{A}_t$ for all $s \geq 0$ and for all $t \geq 0$.
- Let τ be an extended stopping time for the $\mathcal{A}_t$'s.
- Suppose that $P(\tau < \infty) > 0$.
- For $t \geq 0$ define

$$Y(t) = \begin{cases} X(\tau + t) - X(\tau) & \text{on } [\tau < \infty], \\ 0 & \text{on } [\tau = \infty]. \end{cases} \tag{1}$$

Then:

$$Y : ([\tau < \infty] \cap \Omega, [\tau < \infty] \cap \mathcal{A}, P(\cdot|\tau < \infty)) \mapsto (D_{[0,\infty)}, \mathcal{D}_{[0,\infty)})$$

and

$$P(Y \in F|[\tau < \infty]) = P(X \in F) \quad \text{for all} \quad F \in \mathcal{D}_{[0,\infty)}). \tag{2}$$

Moreover, for all $F \in \mathcal{D}_{[0,\infty)})$ and $A \in \mathcal{A}_\tau$

$$P([Y \in F] \cap A|[\tau < \infty]) = P(X \in F) \cdot P(A|[\tau < \infty]). \tag{3}$$

Proof. (Case 1.) Suppose that the range of τ is a countable subset $\{s_1, s_2, \ldots\}$ of $[0, \infty)$. Let $t_1, t_2, \ldots, t_m \geq 0$, let $B_1, \ldots, B_m$ be Borel subsets of $\mathbb{R}$, and let $A \in \mathcal{A}_\tau$. Then

$$\begin{aligned}
&P([Y(t_1) \in B_1, \ldots Y(t_m) \in B_m] \cap A \cap [\tau < \infty]) \\
&\quad = \sum_k P([Y(t_1) \in B_1, \ldots Y(t_m) \in B_m] \cap A \cap [\tau = s_k]) \\
&\quad = \sum_k P([X(t_1 + s_k) - X(s_k) \in B_1, \ldots] \cap A \cap [\tau = s_k]) \\
&\quad = \sum_k P([X(t_1 + s_k) - X(s_k) \in B_1, \ldots]) P(A \cap [\tau = s_k]) \\
&\quad = P([X(t_1) \in B_1, \ldots]) \sum_k P(A \cap [\tau = s_k]) \\
&\quad = P([X(t_1) \in B_1, \ldots, X(t_m) \in B_m]) P(A \cap [\tau < \infty]),
\end{aligned}$$

where the third equality holds since

$$A \cap [\tau = s_k] = (A \cap [\tau \leq s_k]) \cap [\tau = s_k]$$

is in $\mathcal{A}_{s_k}$, and thus independent of the other event by the independent increments of X.

Setting $A = [\tau < \infty]$ in the last display yields

$$
\begin{aligned}
& P(Y(t_1) \in B_1, \ldots, Y(t_m) \in B_m | [\tau < \infty]) \\
& = \; P([X(t_1) \in B_1, \ldots, X(t_m) \in B_m]);
\end{aligned}
$$

Substitution of this equality into the first display of the proof and dividing by τ yields

$$
\begin{aligned}
& P([Y(t_1) \in B_1, \ldots Y(t_m) \in B_m] \cap A|]\tau < \infty]) \\
& = \; P([Y(t_1) \in B_1, \ldots Y(t_m) \in B_m|[\tau < \infty]) \cdot P(A|[\tau < \infty]).
\end{aligned}
$$

Thus the last two displays hold for the class $\mathcal{G}$ of sets of the form $[Y(t_1) \in B_1, \ldots, Y(t_m) \in B_m]$ and for all sets $a \in \mathcal{A}_\tau$. Since $\mathcal{G}$ generates $Y^{-1}(\mathcal{D}_{[0,\infty)})$, the second display of the proof implies (2). Since $\mathcal{G}$ is also closed under finite intersections (that is, it is a $\bar{\pi}-$system), the 3rd display and proposition 7.1.1 imply that the equality in (3) holds.

Case 2: Now consider a general stopping time τ. For $n \geq 1$ define

$$\tau_n = \begin{cases} k/n & \text{for} (k-1)/n < s\tau \leq k/n \ \text{ and } k \geq 1, \\ 1/n & \text{for } \tau = 0, \\ \infty & \text{for } \tau = \infty. \end{cases}$$

Note that $\tau_n(\omega) \searrow \tau(\omega)$ for $\omega \in [\tau < \infty]$. For $k/n \leq t < (k+1)/n$ we have

$$[\tau_n \leq t] = [\tau \leq k/n] \in \mathcal{A}_{k/n} \subset \mathcal{A}_t,$$

so that τ_n is a stopping time, and also for $A \in \mathcal{A}_t$ (so that $\mathcal{A}_\tau \subset \mathcal{A}_{\tau_n}$). Define

$$Y_n(t) = X(\tau_n + t) - X(\tau_n) \ \text{ on } [\tau_n < \infty] = [\tau < \infty],$$

and let it equal 0 elsewhere By case 1, results (b) and (c), both

$$P(Y_n \in F | [\tau < \infty]) = P(X \in F) \ \text{ and} \tag{4}$$

$$P([Y_n \in F] \cap A | [\tau < \infty]) = P(Y_n \in F | [\tau < \infty]) \cdot P(A | [\tau < \infty]). \tag{5}$$

for all $F \in \mathcal{D}_{[0,\infty)}$ and all $A \in \mathcal{A}_\tau$.

Let $(r_1, \ldots, r_m)$ denote any continuity point of the joint df of the finite dimensional random vector $(Y(t_1), \ldots, Y(t_m))$, and define

$$\begin{aligned} G_n &\equiv [Y_n(t_1) < r_1, \ldots, Y_n(t_m) < r_m, \tau < \infty], \\ G &\equiv [Y(t_1) < r_1, \ldots, Y(t_m) < r_m, \tau < \infty], \\ G^* &\equiv [Y(t_1) \leq r_1, \ldots, Y(t_m) \leq r_m, \tau < \infty], \\ H &\equiv [X(t_1) < r_1, \ldots, X(t_m) < r_m]. \end{aligned}$$

By the right-continuity of the sample paths, $Y_n(t) \to Y(t)$ for every t and every $\omega \in [\tau < \infty]$; thus

$$G \subset \liminf G_n \subset \limsup G_n \subset G^*. \tag{6}$$

Then

$$
\begin{aligned}
P(G|\tau<\infty) &\le P(\underline{\lim} G_n|\tau<\infty) \le \underline{\lim} P(G_n|\tau<\infty) \\
&\qquad \text{by (6), then the DCT} \\
&= P(H) = \overline{\lim} P(G_n|\tau<\infty) \quad \text{by using (4) twice,} \\
&\le P(\overline{\lim} G_n|\tau<\infty) \le P(G^*|\tau<\infty) \quad \text{by the DCT and (6)} \\
&\le P(G|\tau<\infty) + \sum_{i=1}^{m} P(Y(t_i)=r_i|\tau<\infty) \\
&= P(G|\tau<\infty)
\end{aligned}
$$

since $(r_1, \ldots, r_m)$ is a continuity point.

Corollary 0. Sums of independent independent and identically distributed random variables are strong-Markov; see PfS, Section 8.7 page 179.

Corollary 1. Standard Brownian motion $\mathbb{S}$ is a strong - Markov process.

Corollary 2. The standard Poisson process $\mathbb{N}$ is a strong - Markov process.

Corollary 3. Markov chains (see e.g. Durrett (2019), section 5.2) are strong - Markov processes.