

## STEIN'S METHOD VIA THE ZERO BIAS TRANSFORMATION

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Let  $X$  be a mean zero random variable with finite non-zero variance  $\sigma^2$ . Let  $X^*$  have the  $X$ -zero biased distribution: that is,  $X^*$  satisfies

$$E[Xf(X)] = \sigma^2 E[f'(X^*)]$$

for all differentiable  $f$  for which  $E[Xf(X)]$  exists.

As Goldstein and Reinert (1997) show, the transformation  $X \mapsto X^*$  (or, more precisely, the transformation  $F \mapsto F^*$  where  $X \sim F$ ,  $X^* \sim F^*$ ) has several important properties. These are summarized in the following lemma.

**Lemma.** The transformation  $X \mapsto X^*$  satisfies the following properties:

- (i)  $X \mapsto X^* = X$  if and only if  $X \sim N(0, \sigma^2)$ .
- (ii) The distribution function  $F^*$  has density  $f^*$  which is unimodal about 0 and is given by

$$f^*(x) = \frac{1}{\sigma^2} \begin{cases} E[X1_{[X>x]}], & x \geq 0, \\ -E[X1_{[X \leq x]}], & x < 0 \end{cases}$$

(Note that since  $E[X] = 0$ ,  $E[X1_{[X>x]}] = -E[X1_{[X \leq x]}]$  for  $x \in \mathbb{R}$ .)

- (iii) If  $X$  is symmetric about 0 (so  $-X \stackrel{d}{=} X$ ), then  $X^*$  is symmetric about 0.

(iv)  $\sigma^2 E[(X^*)^r] = E[X^{r+2}]/(r+1)$  for each integer  $r \in \mathbb{N}$ .

(v) Let  $X_{n,1}, \dots, X_{n,n}$  be independent mean zero random variables with  $E[X_{n,i}^2] = \sigma_{n,i}^2$ . Set  $W_n = \sum_{i=1}^n X_{n,i}$  and  $E[W_n^2] \equiv \sum_{i=1}^n \sigma_{n,i}^2 = 1$ . Let  $I_n$  be a random integer with  $P(I_n = i) = \sigma_{n,i}^2$ . Then  $W_n - X_{n,I_n} + X_{n,I_n}^* \stackrel{d}{=} W_n^*$ .

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(vi) Let  $X$  have mean zero with variance  $\sigma^2$  and distribution function  $F$ . Let  $(\hat{X}, \hat{X}')$  have joint distribution function  $\hat{F}$  given by

$$d\hat{F}(\hat{x}, \hat{x}') = \frac{(\hat{x} - \hat{x}')^2}{2\sigma^2} dF(\hat{x}, \hat{x}').$$

Then, with  $U \sim \text{Uniform}(0, 1)$ ,  $U\hat{X} + (1-U)\hat{X}' \stackrel{d}{=} X^*$ .

As shown by Goldstein (2009), if  $Y \sim \text{Bernoulli}(p)$ , then  $X \equiv Y - p$  (which has  $E[X] = 0$  and  $\text{Var}[X] = p(1-p)$ ) has  $X^* \sim \text{Uniform}(-p, 1-p) \stackrel{d}{=} U - p$  where  $U \sim \text{Uniform}(0, 1)$ . Here we illustrate the transformation via several further examples.

**Example 1.** If  $X \sim \text{Logistic}(0, 1)$  (with  $F(x) = 1/(1 + e^{-x})$ ), then the densities  $f$  and  $f^*$  are as follows:

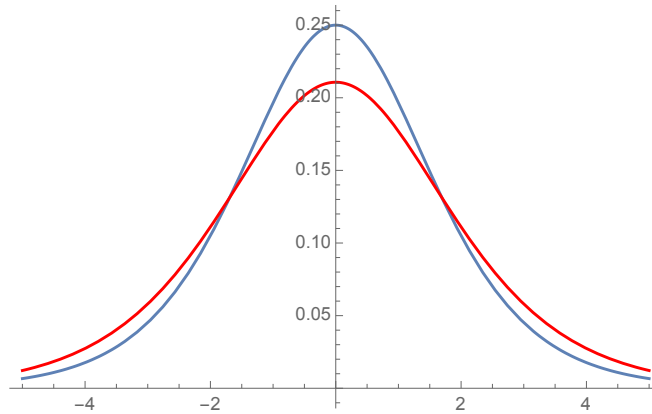


FIG 1. The Logistic density  $f$  (blue) with the zero-bias transform density  $f^*$  (red)

**Example 2.** If  $X \sim$  centered Gumbel distribution (with  $F(x) = 1 - \exp(-e^x)$ ), then the densities  $f$  and  $f^*$  are as follows:

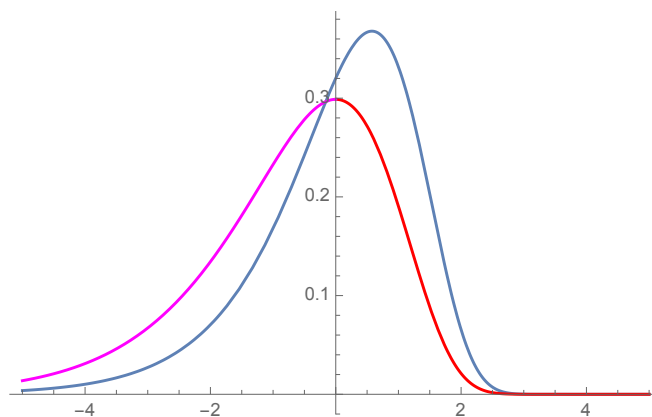


FIG 2. The centered Gumbel density  $f$  (blue) with the zero-bias transform density  $f^*$  (red and magenta)

**Example 3.** If  $X \sim$  centered Weibull(2, 1) distribution (with  $F(x) = 1 - \exp(-x^2)$ ), then the densities  $f$  and  $f^*$  are as follows:

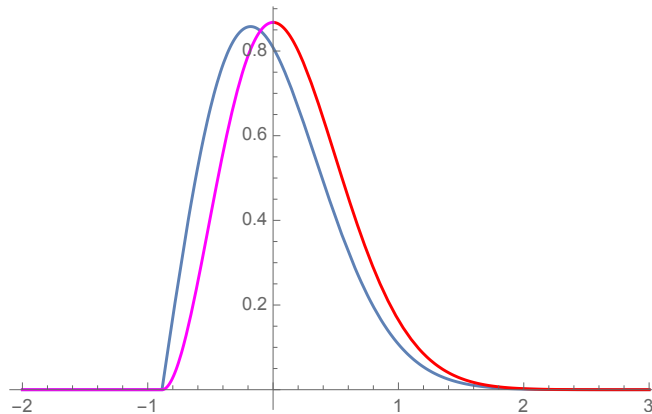


FIG 3. The centered Weibull (2, 1) density  $f$  (blue) with the zero-bias transform density  $f^*$  (red and magenta)

**Example 4.** If  $X \sim$  centered Gamma(4, 1) distribution (with  $f(x) = x^3 \exp(-x)/\Gamma(4)$ ), then the densities  $f$  and  $f^*$  are as follows:

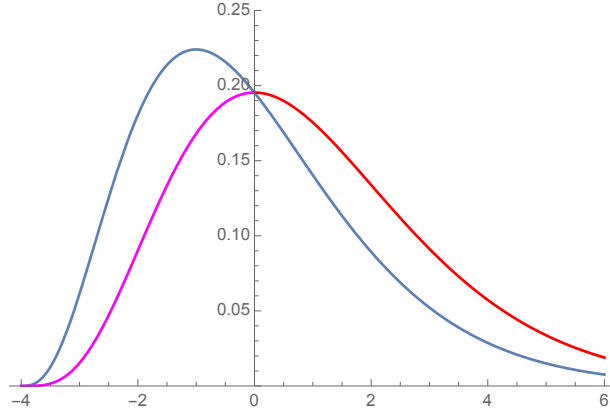


FIG 4. The centered Gamma (4,1) density  $f$  (blue) with the zero-bias transform density  $f^*$  (red and magenta)

**Example 5.** If  $X \sim$  centered Binomial(16, 1/3), then the mass function (and density function)  $f$  and ( $f^*$ ), together with the  $N(0, 16(1/3)(2/3))$  density, are as follows:

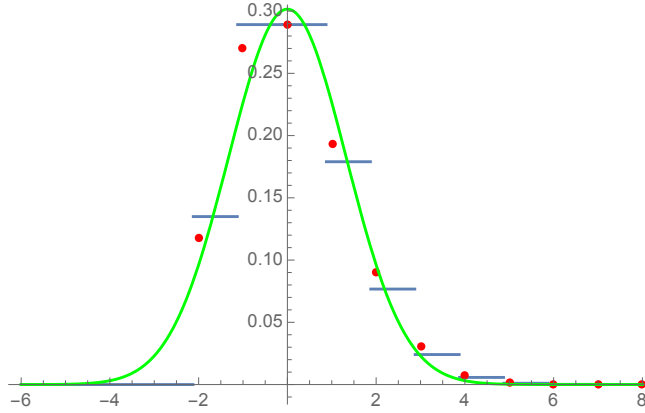


FIG 5. The centered Binomial(16,1) mass function  $f$  (red) with the zero-bias transform density  $f^*$  (blue)

Here are the distribution functions corresponding to the densities in Figure 5:

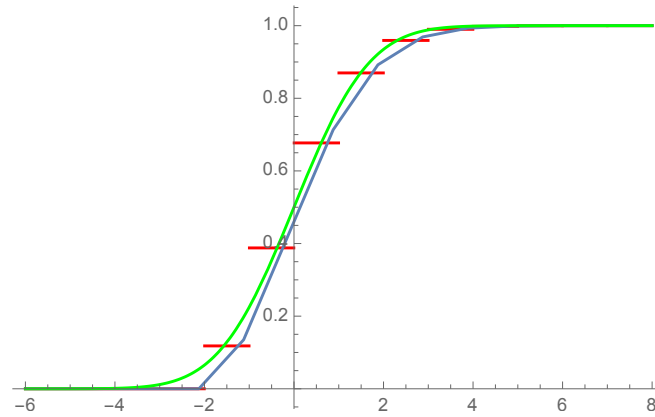


FIG 6. The centered  $\text{Binomial}(16, 1)$  distribution function  $f$  (red) with the zero-bias transform density  $f^*$  (blue), and the  $N(0, 16(1/3)(2/3))$  distribution function (green)

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