

## Statistics 522, Problem Set 7 Solutions

Wellner; 3/7/2020

- Let  $\mathbb{S}$  be standard Brownian motion on  $[0, \infty)$ . For  $b > 0$  fixed, let  $\tau \equiv \tau_b \equiv \inf\{t > 0 : \mathbb{S}(t) = b\}$ . Then  $\tau_b$  is a stopping time. Use the exponential martingale  $Y_r(t) \equiv \exp(r\mathbb{S}(t) - (1/2)r^2t)$  to give a development for  $\tau_b$  parallel to that given in class on 13 February for  $T_b$  in the context of a simple random walk. That is, use optional sampling of the martingale  $Y_r$  to:
  - show that  $P(\tau_b < \infty) = 1$ .
  - Show that the Laplace transform of  $\tau_b$  is given by  $E \exp(-s\tau_b) = \exp(-\sqrt{2sb})$  for  $s \geq 0$ .
  - Show that  $E(\tau_b) = \infty$ .
  - The density of  $\tau_b$  is given by  $f_{\tau_b}(t) = (b/t^{3/2})\phi(b/\sqrt{t})1_{(0,\infty)}(t)$  where  $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$  is the standard normal density. We will show this via reflection arguments for Brownian motion during Spring quarter. Use this expression for the density to show that  $E\tau_b^{1/2} = \infty$  and that  $E\tau_b^r < \infty$  for each  $0 < r < 1/2$ .

**Solution:** For any positive integer  $n$ , consider the new stopping time  $\tau_b \wedge n$ . Since  $Y_c$  is a mean 1 martingale and  $\tau_b \wedge n \leq n$  with  $\mathbb{S}(t) \leq b$  for  $0 \leq t \leq \tau_b$ , it follows by optional sampling that

$$1 = E \exp(r\mathbb{S}(\tau_b \wedge n) - (1/2)r^2(\tau_b \wedge n)).$$

Now  $r\mathbb{S}(\tau_b \wedge n) - (1/2)r^2(\tau_b \wedge n) \leq rb$  for  $r > 0$ , and hence

$$\exp(r\mathbb{S}(\tau_b \wedge n) - (1/2)r^2(\tau_b \wedge n)) \leq \exp(rb).$$

Letting  $n \rightarrow \infty$ , it follows by the dominated convergence theorem that

$$1 = E \exp(r\mathbb{S}(\tau_b) - (1/2)r^2\tau_b) = E \exp(rb - (1/2)r^2\tau_b). \quad (1)$$

Letting  $r \searrow 0$  yields

$$1 = E\{1_{[\tau_b < \infty]}\} = P(\tau_b < \infty).$$

Rearranging (1) yields

$$E \exp(-(1/2)r^2\tau_b) = \exp(-rb). \quad (2)$$

Letting  $s \equiv (1/2)r^2$  this becomes

$$L_b(s) \equiv E \exp(-s\tau_b) = \exp(-\sqrt{2sb}). \quad (3)$$

Differentiating this with respect to  $s$  gives

$$L'_b(s) = \exp(-\sqrt{2sb}) \cdot (-(2s)^{-1/2}b) \stackrel{s=0}{=} +\infty,$$

which implies that  $E\tau_b = \infty$ . Furthermore, the Laplace transform in (3) can be inverted to show that  $\tau_b$  has the claimed density

$$f_{\tau_b}(t) = (b/t^{3/2})\phi(b/\sqrt{t})1_{(0,\infty)}(t). \quad (4)$$

This is more easily derived via reflection arguments for Brownian motion which we will discuss in 523 during Spring quarter. For now we note that for this density

$$\begin{aligned} E\tau_b^{1/2} &= \int_0^\infty t^{1/2} \frac{b}{t^{3/2}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{b^2}{2t}\right) dt \\ &\geq \int_{1/2}^\infty \frac{b}{t} \frac{1}{\sqrt{2\pi}} e^{-b^2} dt = \frac{be^{-b^2}}{\sqrt{2\pi}} \int_{1/2}^\infty t^{-1} dt = \infty. \end{aligned}$$

On the other hand, for  $0 < r < 1/2$  we have  $3/2 - r > 1$  and hence

$$\begin{aligned} E\tau_b^r &= \int_0^\infty t^r \frac{b}{t^{3/2}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{b^2}{2t}\right) dt \\ &\leq \frac{b}{\sqrt{2\pi}} \left\{ \int_0^1 t^{r-3/2} \exp(-b^2/(2t)) dt + \int_1^\infty t^{-(3/2-r)} dt \right\} \\ &\leq \frac{b}{\sqrt{2\pi}} \left\{ \int_1^\infty \exp(-(b^2/2)s) ds + (3/2 - r - 1)^{-1} \right\} \\ &= \frac{b}{\sqrt{2\pi}} \left\{ \exp(-b^2/2) + (3/2 - r - 1)^{-1} \right\} < \infty. \end{aligned}$$

In fact, via Mathematica,

$$E\tau_b^r = \frac{b^{2r}\Gamma(1/2 - r)}{2^r\sqrt{\pi}},$$

and this is confirmed by a direct calculation via the change of variables  $v = b^2/(2t)$ :

$$\begin{aligned}
E\tau_b^r &= \int_0^\infty t^r \frac{b}{t^{3/2}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{b^2}{2t}\right) dt \\
&= \frac{b}{\sqrt{2\pi}} \int_0^\infty t^{r-3/2} \exp\left(-\frac{b^2}{2t}\right) dt \\
&= \frac{b}{\sqrt{2\pi}} \int_0^\infty t^{r+1/2} \exp\left(-\frac{b^2}{2t}\right) t^{-2} dt \\
&= \frac{b}{\sqrt{2\pi}} \int_0^\infty \left(\frac{b^2}{2v}\right)^{r+1/2} \exp(-v) \frac{2}{b^2} dv \\
&= \frac{b}{\sqrt{2\pi}} \left(\frac{b^2}{2}\right)^{r+1/2} \cdot \frac{2}{b^2} \int_0^\infty v^{1/2-r-1} e^{-v} dv \\
&= \frac{b^{2r}}{\sqrt{\pi} 2^r} \Gamma(1/2 - r).
\end{aligned}$$

Confirmation that the density is given correctly by (4) can be accomplished by computing the Laplace transform: the following solution is due to Hongjian Shi:

$$\begin{aligned}
E \exp(-s\tau_b) &= \int_0^\infty e^{-st} \frac{b}{\sqrt{2\pi}} t^{-3/2} \exp\left(-\frac{b^2}{2t}\right) dt \\
&= \frac{b}{\sqrt{2\pi}} \int_0^\infty t^{-3/2} \exp\left(-\left(\frac{b^2}{2t} + st\right)\right) dt \\
&= \sqrt{\frac{A}{\pi}} \int_0^\infty u^{-3/2} \exp\left(-A\left(u + \frac{1}{u}\right)\right) du \quad (5) \\
&\quad \text{by the change of variables } u \equiv (\sqrt{2s}/b)t
\end{aligned}$$

where  $A \equiv b\sqrt{s/2}$ . But by the change of variables  $v = 1/u$  it is seen that

$$\int_0^\infty u^{-3/2} \exp(-A(u + 1/u)) du = \int_0^\infty v^{-1/2} \exp(-A(1/v + v)) dv,$$

and hence we can write

$$\begin{aligned}
& \int_0^\infty u^{-3/2} \exp(-A(u + 1/u)) du \\
&= \int_0^\infty \frac{1}{2} \left( \frac{1}{u^{1/2}} + \frac{1}{u^{3/2}} \right) \exp(-A(u + 1/u)) du \\
&= \int_0^\infty e^{-A(w^2+2)} dw \quad \text{by the change of variables } w \equiv u^{1/2} - u^{-1/2} \\
&= \exp(-2A) \sqrt{\frac{\pi}{A}} \quad \text{by noting that} \\
&\quad \frac{1}{\sqrt{2\pi 2/A}} \int_0^\infty e^{-Aw^2} dw = 1/2.
\end{aligned}$$

Combining this with (5) yields

$$E \exp(-s\tau) = \exp(-2A) = \exp(-2b\sqrt{s}/\sqrt{2}) = \exp(-b\sqrt{2s}).$$

2. PfS Course notes, Exercise 9.1.2 page 195. (PfS 2000, Exercise 11.7.2, page 289). (Pólya's lemma) If  $F_n \rightarrow_d F$  for a continuous df  $F$ , then

$$\|F_n - F\| = \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \rightarrow 0.$$

Thus if  $F_n \rightarrow_d F$  with  $F$  continuous and  $x_n \rightarrow x$ , then  $F_n(x_n) \rightarrow F(x)$ .

**Solution:** Let  $M$  be a (large) positive integer, and set  $x_{M,j} \equiv x_j = F^{-1}(j/(M+1))$  for  $j = 1, \dots, M$ , and let  $x_{M,0} \equiv -\infty$ ,  $x_{M,M+1} \equiv \infty$ . Then for  $x \in [x_{j-1}, x_j]$  we have

$$F_n(x) - F(x) \begin{cases} \leq F_n(x_j) - F(x_{j-1}) \leq F_n(x_j) - F(x_j) + 1/M \\ \geq F_n(x_{j-1}) - F(x_j) \geq F_n(x_{j-1}) - F(x_{j-1}) - 1/M \end{cases},$$

and hence

$$\begin{aligned}
\|F_n - F\| &\leq \max_{1 \leq j \leq M+1} \sup_{x_{j-1} \leq x \leq x_j} |F_n(x) - F(x)| \\
&\leq \max_{1 \leq j \leq M} |F_n(x_j) - F(x_j)| + 1/M \\
&\rightarrow 0 + 1/M
\end{aligned}$$

since  $F_n(x) \rightarrow F(x)$  at all  $x$  in view of  $F$  being continuous. But since  $M$  is arbitrary, this can be made arbitrarily small; i.e.  $\|F_n - F\| \rightarrow 0$ .

The useful corollary is: if  $F_n \rightarrow_d F$  with  $F$  continuous and  $x_n \rightarrow x$ , then  $F_n(x_n) \rightarrow F(x)$ . This follows from Pólya's lemma since

$$\begin{aligned} |F_n(x_n) - F(x)| &\leq |F_n(x_n) - F(x_n)| + |F(x_n) - F(x)| \\ &\leq \sup_{y \in \mathbb{R}} |F_n(y) - F(y)| + |F(x_n) - F(x)| \\ &\rightarrow 0 + 0 = 0 \end{aligned}$$

by continuity of  $F$  for the second term and Pólya's lemma for the first term.

3. PfS Course notes, Exercise 9.1.3 page 195. (PfS 2000, Exercise 11.7.3, page 289). (Verifying tightness) Suppose  $X_n \sim F_n$ . Show that  $\{F_n : n \geq 1\}$  is tight if either: (a)  $\limsup_n E|X_n|^r < \infty$  for some  $r > 0$ ; or (b)  $F_n \rightarrow_d F$ .

**Solution:** (a) Let  $\epsilon > 0$ . Now by Markov's inequality

$$\limsup_{n \rightarrow \infty} P(|X_n| > M) \leq \frac{\limsup_{n \rightarrow \infty} E|X_n|^r}{M^r} < \epsilon$$

for  $M > M(r, \epsilon) \equiv (\limsup_{n \rightarrow \infty} E|X_n|^r / \epsilon)^{1/r}$ . Thus there is an  $N \equiv N(\epsilon, r)$  such that  $\sup_{n > N} P(|X_n| > M) \leq 2\epsilon$ . But since  $F_1, \dots, F_N$  are df's, there exists a  $K = K_\epsilon$  so large that  $\max_{1 \leq n \leq N} P(|X_n| > K) < 2\epsilon$ . Taking  $R = R_\epsilon = \max\{M, K\}$  we have

$$\sup_{1 \leq n < \infty} P(|X_n| > R) < 2\epsilon.$$

Thus  $\{F_n\}$ , the family of distributions of  $\{X_n\}$ , is tight.

(b) Let  $\epsilon > 0$ . Let  $r = r(\epsilon) \in C_F$  be so large that  $1 - F(r) < \epsilon/4$ , and let  $l = l(\epsilon) \in C_F$  be so small that  $F(l) < \epsilon/4$ . Now there exists an  $N = N_\epsilon$  so large that

$$|F_n(r) - F(r)| < \epsilon/4 \quad \text{for all } n > N$$

and

$$|F_n(l) - F(l)| < \epsilon/4 \quad \text{for all } n > N.$$

Then we have

$$\begin{aligned} \sup_{n>N} F_n([l, r]^c) &\leq F(l) + (1 - F(r)) \\ &\quad + |F_n(l) - F(l)| + |F_n(r) - F(r)| \\ &\leq \epsilon/4 + \epsilon/4 + \epsilon/4 + \epsilon/4 = \epsilon \end{aligned}$$

But since  $F_1, \dots, F_N$  are distribution functions, we can easily find an interval  $[l', r']$ , such that

$$\max_{1 \leq n \leq N} F_n([l', r']^c) < \epsilon,$$

and hence the interval  $[a, b] \equiv [l \wedge l', r \vee r']$  satisfies

$$\sup_{1 \leq n < \infty} F_n([a, b]^c) < \epsilon;$$

i.e  $\{F_n\}$  is tight.

4. Exercise 11.6.2, page 34, Wellner, Chapter 11, notes. Suppose that  $\mu_n \rightarrow \mu$  and  $\sigma_n^2 \rightarrow \sigma^2$  where both  $\mu$  and  $\sigma^2$  are finite. Suppose that  $Z \sim P_0$  on  $\mathbb{R}$ .

- (a) Show that  $X_n \stackrel{d}{=} \mu_n + \sigma_n Z \rightarrow_d \mu + \sigma Z \stackrel{d}{=} X$ .  
 (b) Show that for  $f \in BL(\mathbb{R})$

$$|Ef(X_n) - Ef(X)| \leq \|f\|_{BL} E\{1 \wedge (|\mu_n - \mu| + |\sigma_n - \sigma||Z|)\}.$$

**Solution:** (b) For  $f \in BL(\mathbb{R})$  it follows that  $|f(y) - f(x)| \leq \|f\|_{BL} \{1 \wedge |y - x|\}$  for all  $x, y \in \mathbb{R}$ ; recall the inequality before Definition 1.4 on page 4. Thus

$$|f(\mu_n + \sigma_n Z) - f(\mu + \sigma Z)| \leq \|f\|_{BL} \{1 \wedge |\mu_n - \mu + (\sigma_n - \sigma)Z|\};$$

This implies that

$$\begin{aligned} |Ef(X_n) - Ef(X)| &= |Ef(\mu_n + \sigma_n Z) - f(\mu + \sigma Z)| \\ &\leq E|f(\mu_n + \sigma_n Z) - f(\mu + \sigma Z)| \\ &\leq \|f\|_{BL} E\{1 \wedge |\mu_n - \mu| + |\sigma_n - \sigma||Z|\}. \end{aligned}$$

- (a) Since  $\mu_n - \mu \rightarrow 0$  and  $\sigma_n - \sigma \rightarrow 0$ , it follows by the dominated convergence theorem (with dominating function 1) that the right side of the last display converges to 0. Thus  $Ef(X_n) \rightarrow Ef(X)$  for all  $f \in BL(\mathbb{R})$ ; therefore  $X_n \rightarrow_d X$  by the portmanteau theorem.