Statistics 522, Problem Set 6 Solutions

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1. Polyá's urn: At time 0, an urn contains 1 black ball and 1 white ball. At each time $1, 2, 3, \ldots$, a ball is chosen at random from the urn, and is replaced together with a new ball of the same color. Just after time n, there are therefore $n + 2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. Let $M_n = (B_n + 1)/(n + 2)$, the proportion of black balls in the urn just after time n. Prove that (relative to a natural filtration which you should specify) M_n is a martingale. Prove that $P(B_n = k) = 1/(n+1)$ for $0 \le k \le n$. What is the distribution of $\Theta \equiv \lim_n M_n$? Prove that for $0 < \theta < 1$,

$$N_n^\theta \equiv \frac{(n+1)!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}$$

defines a martingale N_n^θ.

Solution: Let $\mathcal{F}_n \equiv \sigma(B_1, \ldots, B_n)$. Note that $M_n \equiv (B_n + 1)/(n + 2)$ is the conditional (given $\mathcal{F}_n$) probability of drawing a black ball at the $n + 1$st draw. Thus we compute

$$\begin{aligned}
E(M_{n+1}|\mathcal{F}_n) &= E\left(\frac{B_{n+1}+1}{n+3}|\mathcal{F}_n\right) = \frac{1}{n+3}E(B_{n+1}+1|\mathcal{F}_n) \\
&= \frac{1}{n+3}\{(B_n+1)(1-M_n) + (B_n+2)M_n\} \\
&= \frac{1}{n+3}\{B_n + 1 - M_n + 2M_n\} \\
&= \frac{1}{n+3}\{(n+2)M_n + M_n\} = M_n \qquad \text{a.s.}\,.
\end{aligned}$$

Hence $\{M_n, \mathcal{F}_n\}$ is a martingale. Similarly, letting

$$p_n(k) \equiv \frac{(n+1)!}{k!(n-k)!}\theta^k(1-\theta)^{n-k},$$

the process $N_n^\theta = p_n(B_n)$ and

$$\begin{aligned} E(N_{n+1}^\theta|\mathcal{F}_n) &= E(p_{n+1}(B_{n+1})|\mathcal{F}_n) \\ &= p_{n+1}(B_n)(1-M_n) + p_{n+1}(B_n+1)M_n \\ &= \frac{(n+2)!}{B_n!(n+1-B_n)!}\theta^{B_n}(1-\theta)^{n+1-B_n}\frac{(n+1-B_n)}{(n+2)} \\ &\qquad + \frac{(n+2)!}{(B_n+1)!(n+1-B_n-1)!}\theta^{B_n+1}(1-\theta)^{n+1-B_n-1}\frac{(B_n+1)}{(n+2)} \\ &= \frac{(n+1)!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}\{(1-\theta)+\theta\} \\ &= p_n(B_n) \equiv N_n^\theta \qquad \text{a.s.}, \end{aligned}$$

so $\{N_n^\theta, \mathcal{F}_n\}$ is a martingale. This implies that $EN_n^\theta = EN_0^\theta = 1$ for all $\theta \in (0,1)$, or

$$E\left\{\frac{n!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}\right\} = \frac{1}{n+1}\,. \tag{1}$$

This equality clearly holds if $P(B_n = k) = 1/(n+1)$ for $k = 0, \ldots, n$. On the other hand, (1.1) implies, by letting $\alpha = \theta/(1-\theta)$, that, with $p_k = P(B_n = k)$,

$$\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\alpha^k p_k = \frac{1}{n+1}(1+\alpha)^n = \frac{1}{n+1}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\alpha^k\,,$$

and this yields $p_k = 1/(n+1)$ for $k = 0, \ldots, n$ by matching coefficients.

The distribution of B_n is a discrete uniform distribution on $0, \ldots, n$ for every n, so the distribution of M_n is a discrete uniform distribution on $0 < 1/(n+2) < \cdots < (n+1)/(n+2) < 1$ and it is clear that $M_n \to_d U(0,1)$ as $n \to \infty$; $P(M_n \le u) = [(n+2)u]/(n+1) \to u = P(U \le u)$ where $U \sim \text{Uniform}(0,1)$.

2. Exercise 13.3.7, PfS Course Notes, page 359. [Exercise 18.3.7, PfS (2000), page 477.] Let $r > 1$. Let $\{X_n, \mathcal{A}_n\}_{n=0}^\infty$ be a martingale. Then the following are equivalent:
(10) The $|X_n|^r$−process is integrable.
(11) $X_n \to_r X_\infty$

(12) The X_n's are uniformly integrable (thus $X_n \to$ (some X_∞) a.s.) and $X_\infty \in L_r$.
(13) The $|X_n|^r$'s are uniformly integrable.
(14) $\{|X_n|^r, \mathcal{A}_n\}_{n=0}^{\infty}$ is a submg and $E|X_n|^r \nearrow E|X_\infty|^r$.

Solution: Suppose that (10) holds. Then $|X_n|^r$ is an integrable submg. Thus the $|X_n|^r$ are uniformly integrable by problem 5, Problem Set # 5. Thus (13) holds.
Suppose (13) holds. Then $\{X_n\}$ is uniformly integrable, and $X_n \to_{a.s.} X_\infty \in L_1$ and

$$E|X_\infty|^r = E(\liminf |X_n|^r) \le \liminf E|X_n|^r \le \sup_n E|X_n|^r < \infty\,,$$

so $X_\infty \in L_r$; i.e. (12) holds.
Suppose (12) holds. Then $\{|X_n|, \mathcal{A}_n\}_{n=0}^{\infty}$ is a sub-martingale by Theorem 16.3.1. Thus $|X_n| \le E(|X_\infty|\mathcal{A}_n)$, so $|X_n|^r \le \{E(|X_\infty||\mathcal{A}_n)\}^r \le E(|X_n|^r|\mathcal{A}_n)$ a.s., and hence $E|X_n|^r \le E|X_\infty|^r < \infty$; i.e. (10) holds.
Thus (10) iff (12) iff (13) holds.
Now (11) implies (10) since

$$E|X_n|^r \le c_r\{E|X_n - X_\infty|^r + E|X_\infty|^r\}$$

by the c_r-inequality.
Suppose that (13) holds. Then $X_n \to_{a.s.} X_\infty \in L_r$ (by (13) implies (12)), and since $\{|X_n|^r, \mathcal{A}_n\}_{n=0}^{\infty}$ is a sub-mg,

$$\limsup_{n\to\infty} E|X_n|^r \le E|X_\infty|^r < \infty\,.$$

Hence $X_n \to_r X_\infty$ by Vitali's theorem; i.e. (11) holds. Thus (10) iff (12) iff (13) iff (14).

3. Suppose that $X_1, X_2, \ldots$ are independent random variables on $(\Omega, \mathcal{A})$ and that X_n has density p_n or q_n under P or Q respectively where p_n and q_n are (for simplicity) everywhere positive on $\mathbb{R}$. Let $\mathcal{F} = \sigma[X_1, X_2, \ldots]$ and $\mathcal{F}_n = \sigma[X_1, \ldots, X_n]$ for $n \ge 1$. Let $Y_n \equiv q_n(X_n)/p_n(X_n)$.
(a) Show that

$$M_n \equiv \frac{dQ}{dP}\Big|_{\mathcal{F}_n} = Y_1 \cdots Y_n$$

where the Y_n's are independent and have mean 1 under P; Hence the likelihood ratio martingale of Example 1.14 is the Kakutani product martingale of Example 1.15.
(b) Show that Q is absolutely continuous relative to P on $\mathcal{F}$ if and only if the martingale $\{M_n, \mathcal{F}_n\}$ is uniformly integrable.
(c) Conclude from Kakutani's theorem (PfS Example 4.4, pages 482-483) that $Q \ll P$ on $\mathcal{F}$ if and only if

$$\prod_{n=1}^{\infty} E(Y_n^{1/2}) = \prod_{n=1}^{\infty} \int_{\mathbb{R}} \sqrt{p_n(x) q_n(x)} dx > 0 \,.$$

(d) Construct two examples of sequences p_n and q_n, one in which the condition in (c) holds and one in which it fails. What is the statistical meaning when it holds and when it fails?

Solution: (a) Let $A_i \in \sigma(X_i)$ for $i = 1, \ldots, n$. Then

$$\begin{aligned}
E_P\{1_{A_1 \times \cdots \times A_n} \frac{dQ}{dP}\} &= E_Q\{1_{A_1 \times \cdots \times A_n}\} \\
&\qquad \text{by definition of the Radon-Nikodym derivative} \\
&= \prod_{i=1}^{n} E_Q(1_{A_i}) \qquad \text{by independence} \\
&= \prod_{i=1}^{n} \int_{\mathbb{R}} 1_{A_i}(x) q_i(x) d\mu(x) \qquad \text{by existence of the densities } q_i \\
&= \prod_{i=1}^{n} \int_{\mathbb{R}} 1_{A_i}(x) \frac{q_i(x)}{p_i(x)} p_i(x) d\mu(x) \\
&= \prod_{i=1}^{n} E_P\{1_{A_i} Y_i\} \\
&= E_P\{1_{A_1 \times \cdots \times A_k} Y_1 \cdots Y_n\} \qquad \text{by independence}\,.
\end{aligned}$$

Now $Y_1 \cdots Y_n$ is $\mathcal{F}_n$ measurable (since it is a function of $X_1, \ldots, X_n$ and agrees with $dQ/dP|_{\mathcal{F}_n}$ on the $\overline{\pi}-$ system $\sigma(X_1) \times \cdots \times \sigma(X_n)$. Thus the claimed equality holds. The Y_i's are independent because the X_i's are independent and they have mean 1 because

$$E_P Y_i = \int_{\mathbb{R}} \frac{q_i(x)}{p_i(x)} p_i(x) d\mu(x) = \int_{\mathbb{R}} q_i(x) dx = 1 \,.$$

(b) If $Q \ll P$, with Radon-Nikodym derivative $dQ/dP \equiv Z$, then $M_n = E(Z|\mathcal{F}_n)$ with $E_P(Z) = Q(\mathbb{R}^\infty) = 1$, so $\{M_n, \mathcal{F}_n\}_{n=0}^{\overline{\infty}}$ is a martingale closed at infinity and is uniformly integrable. Conversely, if $\{M_n\}$ is uniformly integrable, then $M_n \to_{a.s.} M_\infty$ and $E(M_\infty|\mathcal{F}_n) = M_n$ almost surely for every n. Now consider the measures Q and $\tilde{Q}$ defined by

$$\tilde{Q}(A) = E\{1_A M_\infty\}\,.$$

These measures agree on the π-system $\cup\mathcal{F}_n$, and hence they agree on $\mathcal{F} = \sigma(\cup\mathcal{F}_n)$. This implies (by the pi-lambda theorem) that Q and $\tilde{Q}$ agree on $\mathcal{F}$, and hence $M_\infty = dQ/dP$ on $\mathcal{F}$, and $Q \ll P$.

(c) By Kakutani's theorem we conclude that Q is absolutely continuous with respect to P on $\mathcal{F}$ if and only if

$$\prod_{n=1}^{\infty} E(Y_n^{1/2}) = \prod_{n=1}^{\infty} \int_{\mathbb{R}} (q_n(x)/p_n(x))^{1/2} p_n(x)dx = \prod_{n=1}^{\infty} \int_{\mathbb{R}} \sqrt{p_n(x)q_n(x)}dx > 0\,.$$

Since this is symmetric in p_n and q_n and since these densities are everywhere positive, we also can conclude that P is absolutely continuous with respect to Q on $\mathcal{F}$; thus Q and P are mutually absolutely continuous or *equivalent* on $\mathcal{F}$.

(d) Suppose that $p_n(x) = \exp(-x)1_{[0,\infty)}(x)$ and $q_n(x) = \lambda_n \exp(-\lambda_n x)1_{[0,\infty)}(x)$ with $\lambda_n = 1 + c_n$ where $c_n \to 0$. Then we compute

$$E(Y_n^{1/2}) = \int_0^\infty \lambda_n^{1/2} \exp(-(1+\lambda_n)x/2)dx = \frac{2\lambda_n^{1/2}}{1+\lambda_n}\,,$$

and

$$\begin{aligned} H^2(P_n, Q_n) &= \frac{1}{2}\int (\sqrt{p_n(x)} - \sqrt{q_n(x)})^2 dx = 1 - E(Y_n^{1/2}) \\ &= 1 - \frac{2\lambda_n^{1/2}}{1+\lambda_n} \\ &= \frac{1+\lambda_n - 2\lambda_n^{1/2}}{1+\lambda_n} \\ &= \frac{2+c_n-2(1+c_n)^{1/2}}{2+c_n} \\ &\sim \frac{1}{8}c_n^2 \qquad \text{as} \quad n\to\infty \end{aligned}$$

since $c_n \to 0$ and $(1+c_n)^{1/2} = 1 + (1/2)c_n - (1/8 + o(1))c_n^2$. Thus if $c_n = n^{-r}$ with $r > 1/2$ it follows that

$$\sum_1^\infty (1 - E(Y_n^{1/2})) = \sum_1^\infty H^2(P_n, Q_n) < \infty\,,$$

and $Q \ll P$ on $\mathcal{F}$. If $c_n = n^{-1/2}$, then

$$\sum_1^\infty (1 - E(Y_n^{1/2})) = \sum_1^\infty H^2(P_n, Q_n) = \infty\,,$$

and by Kakutuni's theorem we conclude that $M_\infty = 0$ almost surely P. In this case Q and P are singular on $\mathbb{R}^\infty$: there is a set $A \subset \mathbb{R}^\infty$ such that $Q(A) = 1$ and $P(A) = 0$; i.e. $P(A^c) = 1$.

4. Suppose that X and Y are non-negative random variables which satisfy the following inequality:

$$\lambda P(X \ge \lambda) \le E(Y 1_{[X \ge \lambda]}) \quad \text{for every} \quad \lambda > 0. \tag{2}$$

(i) Let $p > 1$ and suppose that $0 < E(X^p) < \infty$ and $E(Y^p) < \infty$. Show that the inequality in the last display implies that

$$\|X\|_p^p = E(X^p) \le \left(\frac{p}{p-1}\right)^p E(Y^p). \tag{3}$$

(ii) Can you relax the assumption $E(X^p) > 0$ or the assumption $E(Y^p) < \infty$?

Solution: (i) Using $E(X^p) = \int_0^\infty p\lambda^{p-1} P(X \ge \lambda) d\lambda$ followed by the hypothesized inequality (2) yields

$$\begin{aligned}
E(X^p) &= \int_0^\infty p\lambda^{p-1} P(X \ge \lambda) d\lambda = \int_0^\infty p\lambda^{p-2} \lambda P(X \ge \lambda) d\lambda \\
&\le \int_0^\infty p\lambda^{p-2} E\{Y 1_{[X \ge \lambda]}\} d\lambda \quad \text{by (2)} \\
&= E\left\{ Y \int_0^\infty p\lambda^{p-2} 1_{[X \ge \lambda]} d\lambda \right\} \quad \text{by Tonelli-Fubini} \\
&= \frac{p}{p-1} E\left\{ Y X^{p-1} \right\} \\
&\le \frac{p}{p-1} [E(Y^p)]^{1/p} \cdot [E(X^p)]^{(p-1)/p} \quad \text{by Hölder's inequality.}
\end{aligned}$$

If $0 < E(X^p) < \infty$ this yields

$$[E(X^p)]^{1/p} \leq \left(\frac{p}{p-1}\right) [E(Y^p)]^{1/p}.$$

or, equivalently,

$$E(X^p) \leq \left(\frac{p}{p-1}\right)^p E(Y^p). \tag{4}$$

(ii) Note that $E(X^p) \geq 0$ since $X \geq 0$ a.s.. If $E(X^p) = 0$, then $X = 0$ a.s. and the inequality (3) holds trivially. If $E(Y^p) = \infty$, then (3) also holds trivially. Thus we may assume that $E(X^p) > 0$ and $E(Y^p) < \infty$. To relax the assumption $E(X^p) < \infty$, replace X by $X \wedge n$. Then since $\{X \wedge n \geq \lambda\}$ is either $\{X \geq \lambda\}$ or $\emptyset$, the hypothesized inequality (2) continues to hold with X replaced by $X \wedge n$. Applying the result derived above assuming $E(X^p) < \infty$ yields

$$E((X \wedge n)^p) \leq \left(\frac{p}{p-1}\right) [E(Y^p)]^{1/p}.$$

Then application of the monotone convergence yields (4), and finiteness of $E(X^p)$ follows from finiteness of $E(Y^p)$.